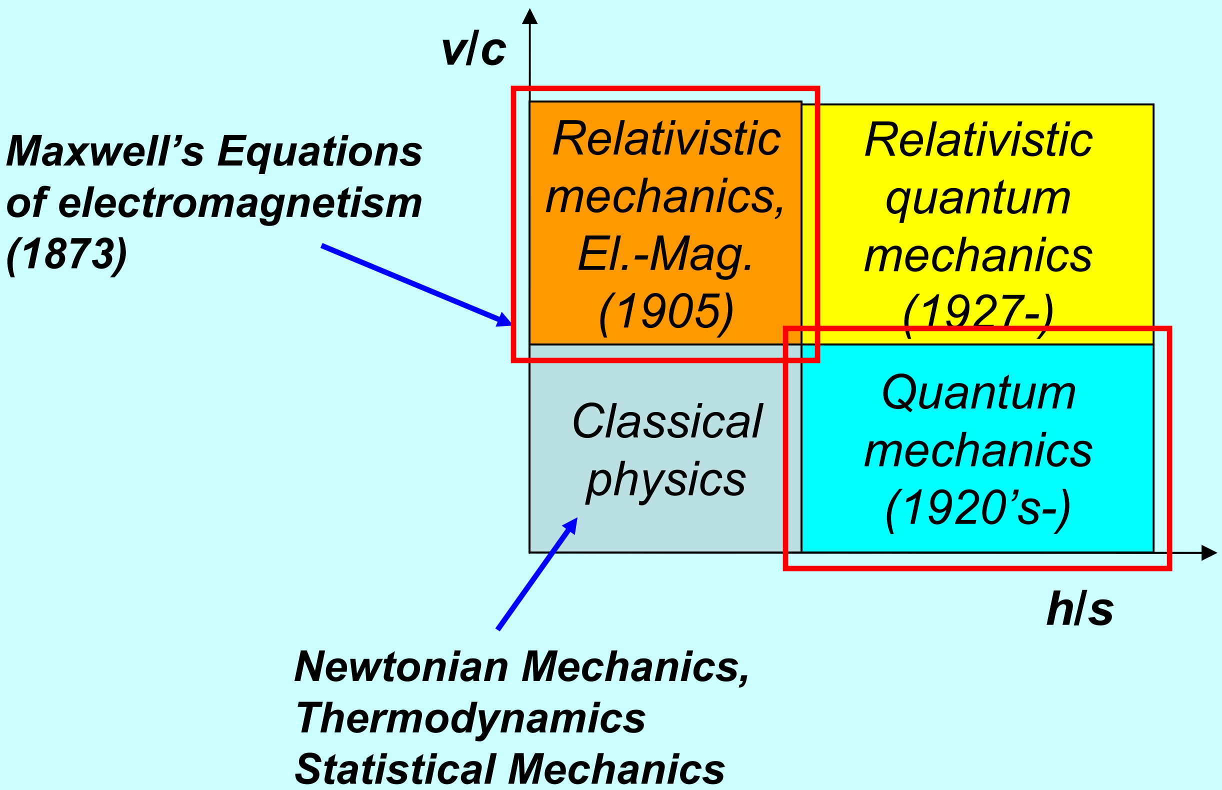
# *Elements of Relativistic Kinematics & Dynamics*

Content:

- ***Classical mechanics***
- ***The relativity principle***
- ***Lorentz transformation***
    - ***4-vectors***
    - ***covariant and contravariant components***
    - ***reference frames***
    - ***Minkowski metric, rapidity***
- ***Two-body decays***
- ***Three-body decays***
- ***Particle collisions***
    - ***elastic collisions***
    - ***inelastic collisions: quasi-elastic collisions, particle creation***
- ***Deep inelastic scattering***
- ***Phase space***
- ***Dynamics – collective expansion in heavy ion collision***

# Modern Physics "Landscape"




**S – the action=momentum×distance, units g×cm²/s**

# *Galilean Principle of Relativity*

***Reference frame*** *– a set of 3 spatial coordinates (e.g.* ***x,y,z****) and a time coordinate* ***t****.*

***Space and Time in Classical Mechanics:***

***Space*** *–* ***uniform and isotropic****.*
***Time*** *–* ***uniform and absolute*** (*the same in both IRFs and non-IRFs*).

***Inertial reference frame (IRF)***
(inspired by experimental observation & their existence is postulated by the 1st Newton's Law):
***it's the frame where bodies removed from interaction with other bodies will maintain their state of rest or of uniform straight-line motion***.

***Practical definition of IRF***:
***a frame that moves with a constant velocity relative to very distant objects, e.g. distant stars.***

***Significance of IRFs:***
Newton's Laws have the same form in all these frames
(the laws are ***invariant*** under the coordinate transformations that transform one IRF into another one).

**Galilean Principle of Relativity (1632) "The laws of classical mechanics are invariant in all inertial reference frames"**

# *Newton's Laws (1687)*

- ## *First law*

It is possible to select a set of ***reference frames***, called ***inertial reference frames***, in which a particle moves without any change in ***velocity*** if no net ***force*** acts on it.
This law is often simplified into the sentence: ***"A particle will stay at rest or continue to move at a constant velocity unless an external unbalanced force acts on it."***

- ## *Second law*

Observed from an inertial reference frame, the net force on a particle is proportional to the time rate of change of its linear ***momentum***: $F = d\,(mv)\,/\,dt$. Momentum $mv$ is the product of ***mass*** and ***velocity***. ***Force*** and ***momentum*** are **vector** quantities and the ***resultant force*** is found from all the forces present by ***vector addition***.
This law is often stated as: **"*$F = ma$*: *the net force on an object is equal to the mass of the object multiplied by its acceleration."***

- ## *Third law*

Whenever a particle ***A*** exerts a force on another particle ***B***, ***B*** simultaneously exerts a force on ***A*** with the same magnitude in the opposite direction. The strong form of the law further postulates that these two forces act along the same line.
This law is often simplified into the sentence ***"Every action has an equal and opposite reaction."***

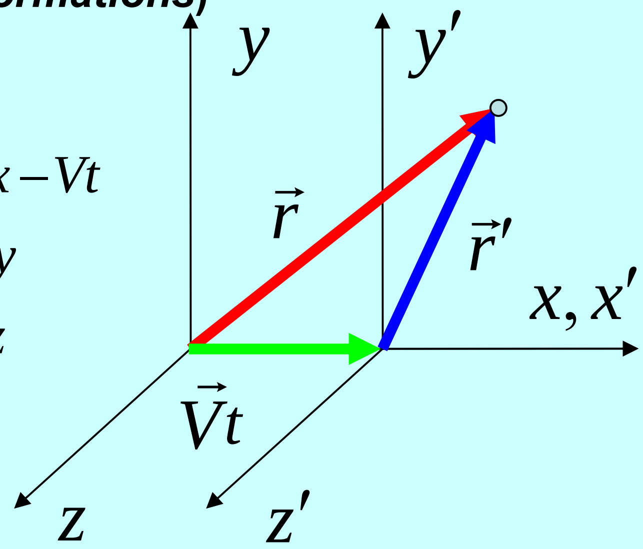
# Galilean transformations

***Galilean Transformations -*** *The IRF transformations that preserve "invariance" of Newton's Laws*

***(Newton Laws are invariant under Galilean transformations)***

$$\begin{cases} t' = t & \text{- the "absolute" time} \\ \vec{r}' = \vec{r} - \vec{V}t & \text{-because the space is uniform and isotropic, the IRFs can move one relative to the other with constant velocity} \end{cases}$$

$$t' = t$$
$$x' = x - Vt$$
$$y' = y$$
$$z' = z$$




(at $t$=0 the origins coincide with one another)

Let's check that the 2nd Newton's Law is invariant under Galilean Transformations:

$$\frac{d\vec{r}'}{dt'} = \frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} - \vec{V} \quad \vec{v}' = \vec{v} - \vec{V} \quad \Rightarrow$$ ***Galilean velocity addition rule***

$$\frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt} \quad \vec{a}' = \vec{a} \Rightarrow$$ acceleration is the same in all IRFs

$\Rightarrow$ The force in Newton mechanics can depend (only!) on the ***difference*** of two radius-vectors and velocities.

$$\Rightarrow \quad \vec{F}' = \vec{F}$$

**!** (This is ***not*** the case in non-inertial RFs***, i.e.*** of an accelerated RF)

# *Importance of vectors in classical mechanics*

***Invariant:*** Let's formulate a physical law as A=B.
If a coordinate transformation affects neither A nor B, we say that this law is ***invariant*** under the transformation.

Galilean transformation do not affect the length of a vector

$\Rightarrow$ ***the length of a vector is invariant under Galilean transformation***

$$t' = t \quad x' = x - Vt \quad y' = y \quad z' = z \quad \Rightarrow$$

$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$length' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2} = length$$

$\Rightarrow$if one can formulate a law that looks like **"vector" = "vector"**, it automatically means that this law is invariant under Galilean transformation

All laws of classical physics must have the following forms to be invariant under Galilean transformation:

***"scalar A" = "scalar B"*** ***"vector A" = "vector B"***

$\Rightarrow$ if one side of an equation is a scalar (vector), the other side must also be a scalar (vector) to satisfy ***Galilean Principle of Relativity***.

# *Maxwell's Equations: challenge to Galilean PR*

$$\nabla \cdot \vec{E} = \rho / \varepsilon_0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}$$

In 1873, Maxwell formulated Equations of Electromagnetism.

- Maxwell's Equations ***describe very well all observed electromagnetic phenomena***
- they are ***not invariant*** under Galilean transformations

!?

***Few remarks:***

- there is a built-in asymmetry: a charge in motion produces a magnetic field while a charge at rest does not
- it follows from Maxwel equations that ***the speed of light is the same in all IRFs*** at odds with Galilean velocity addition.

This asymmetry triggered an idea of a unique stationary RF (***ether***), with respect to which all velocities have to be measured, and where Maxwell equations can be written in their usual form.

However, the famous ***Michelson-Morley experiment*** (1887) did not detect any motion of the Earth with respect to the ether.

***How one could proceed?***

At least one of the following statements ***must be wrong***:

(a) the principle of relativity applies to both mechanical and electromagnetic phenomena

(b) Maxell equations are correct

(c) Galilean transformations are correct

# *Einstein's Principle of Relativity*

***Einstein (1905) assumed that (a) and (b) are correct & postulated:***

***The first postulate of the Special Theory of Relativity:***
***"The laws of physics are the same (covariant) in all IRFs"***

**Covariance is less restrictive** than **invariance**:
Let **A=B**.
If, **under RF transformation, both A and B are transformed into A' and B'**, but still **A'=B'**, **than the law is covariant**.

**One of the consequence** of Einstein's Principle of Relativity (being applied to Maxwell's Equations): **the speed of light in vacuum is the same in all IRFs** and **doesn't depend on the motion of the source of light or an observer**
(in line with the experimental evidence that the ether does not exist).

**However, this applies to all** (not necessarily e.-m.) **phenomena**.
Therefore:

***The second postulate:***
***"The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source"***

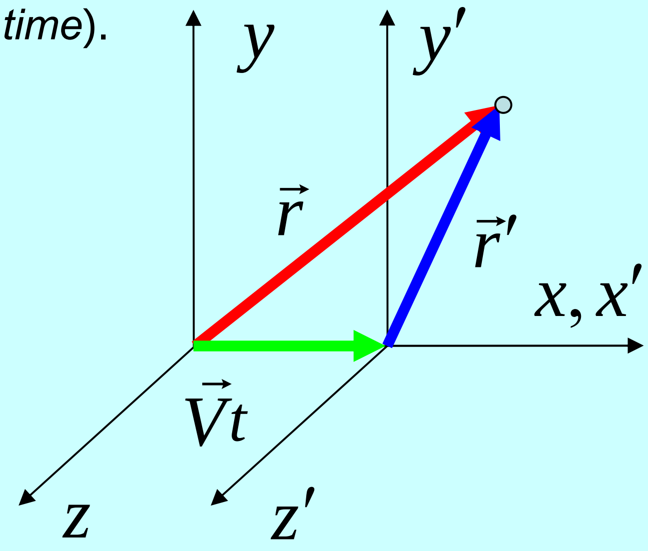
$\Rightarrow$ Maxwell's Equations are in line with Einstein's Principle of Relativity.

***Conclusion:***

- ***Galilean Transformations based on the idea of universal and absolute time are wrong***!
- **Consequently, the laws of mechanics have to be modified to be covariant under new (correct) transformations**.

# Lorentz Transformations

**The class of transformations that maintain the covariance of Maxwell's equations were derived by Lorentz (1904)** (*though Lorentz suggested that the ether wind physically compresses all matter in just the right way to conceal the variations of **c** in Michelson-Morley experiment, he still believed in absolute time*).




- let's consider two inertial frames **S & S'**

**S :** Origin **O**, coordinates **x,y,z** and time **t**
**S′:** Origin **O′**, coordinates **x′,y′,z′** and time **t′**

**S′** *moves with constant velocity along* **x** *axis of* **S**
at time **t=t′=0** the origins **O** and **O′** coincide

∗ ***at t=0 a light flash is produced in O***
⇒ *a spherical wave with radius **R** develops in frame **S***
*a spherical wave with radius **R'** develops in frame **S'***

$$R = c \cdot t \rightarrow x^2 + y^2 + z^2 = (ct)^2$$
$$\rightarrow (ct)^2 - x^2 - y^2 - z^2 = 0$$

$$R' = c \cdot t' \rightarrow x'^2 + y'^2 + z'^2 = (ct')^2$$
$$\rightarrow (ct')^2 - x'^2 - y'^2 - z'^2 = 0$$

$\Rightarrow$ $\left[(c \cdot t)^2 - radius^2\right]$ *is the same in both frames*

# Lorentz Transformations

$$\begin{cases} y' = y \\ z' = z \end{cases} \quad \Rightarrow \quad (ct)^2 - x^2 = (ct')^2 - (x')^2 \rightarrow (x')^2 - x^2 = c^2\left[(t')^2 - t^2\right] \qquad (1)$$

*laws of nature are the same in* **S** *and* **S'** $\Rightarrow$ ***x'*** *and* ***t'*** *are linear functions of* ***x*** *and* ***t***

$$\Rightarrow \begin{cases} x' = \alpha_1 x + \alpha_2 t \\ t' = \alpha_3 x + \alpha_4 t \end{cases}$$

$$if \quad x' = 0 \quad x = -\frac{\alpha_2}{\alpha_1} t \quad \rightarrow \quad \frac{dx}{dt} = -\frac{\alpha_2}{\alpha_1} \equiv v$$

$$if \quad x = 0 \quad \begin{cases} x' = \alpha_2 t \\ t' = \alpha_4 t \end{cases} \quad \rightarrow \quad \frac{dx'}{dt'} = \frac{1}{\alpha_4}\frac{dx'}{dt} = \frac{\alpha_2}{\alpha_4} \equiv -v$$

$$\Rightarrow \alpha_4 = \alpha_1 \quad \& \quad \alpha_2 = -v\alpha_1$$

$$\Rightarrow \begin{cases} x' = \alpha_1 x - v\alpha_1 t \\ t' = \alpha_3 x + \alpha_1 t \end{cases} \qquad (2)$$

# Lorentz Transformations

(1) & (2) $\Rightarrow \quad (\alpha_1^2-1)x^2-2v\alpha_1^2xt+v^2\alpha_1^2t^2 = c^2\alpha_3^2x^2+2c^2\alpha_3\alpha_1xt+c^2(\alpha_1^2-1)t^2 \qquad \forall \quad (x\&t)$

$$\Rightarrow \begin{cases} (\alpha_1^2-1) = c^2\alpha_3^2 \\ -2v\alpha_1^2 = 2c^2\alpha_3\alpha_1 \\ v^2\alpha_1^2 = c^2(\alpha_1^2-1) \end{cases}$$

$$\Rightarrow \alpha_1^2 = \frac{1}{1-\frac{v^2}{c^2}} \quad \to \quad \alpha_1 = \frac{\pm 1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$if \quad v=0 \quad \alpha_1 = \pm 1$ $\qquad$ $but \quad v=0 \quad \to \quad no\ movement\ of\ S'$

$$\to \quad x' = x \quad \to \quad \alpha_1 = 1 \quad \Rightarrow \quad \alpha_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

***Final result:***

$$\begin{cases} x' = \gamma(x-vt) \\ y' = y \\ z' = z \\ t' = \gamma(t-\frac{\beta}{c}x) \end{cases}$$

***(Lorentz Transformations)***

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

# *Lorentz Transformations*

Linearity of Lorentz transformations reflect the fact that:

- the space is uniform and isotropic
- the time – uniform.

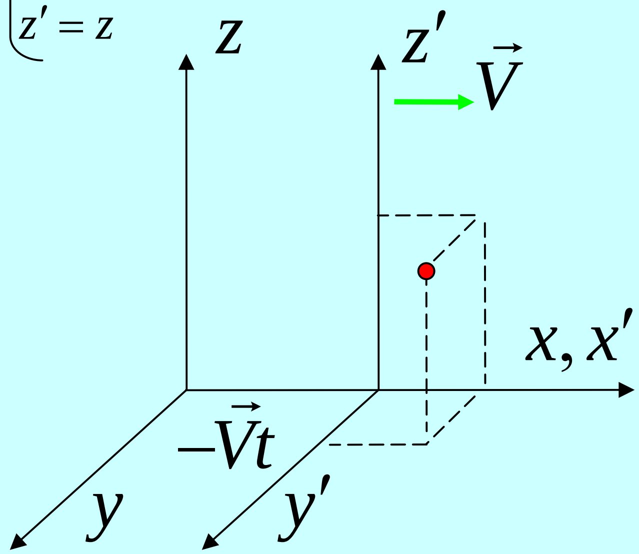
For small *V*<<c ($\beta$<<1, $\gamma$~1)
– Lorentz transformations are reduced to Galilean transformations:

$$\begin{cases} ct' = ct \\ x' = x - \beta ct = x - Vt \\ y' = y \\ z' = z \end{cases}$$

**Example:**

A flash of light occurs at ***x*** = 1m, ***y*** = 1m, ***z*** = 1m, and ***ct*** = 1m (so ***t*** = $3.3\times10^{-9}$s).
Locate this event in the primed RF, which moves at ***V/c***=0.6 to the right.

$\beta$ =0.6, $\gamma$ =1.25, so $\beta\gamma$ =0.75




**Galilean Tr.**

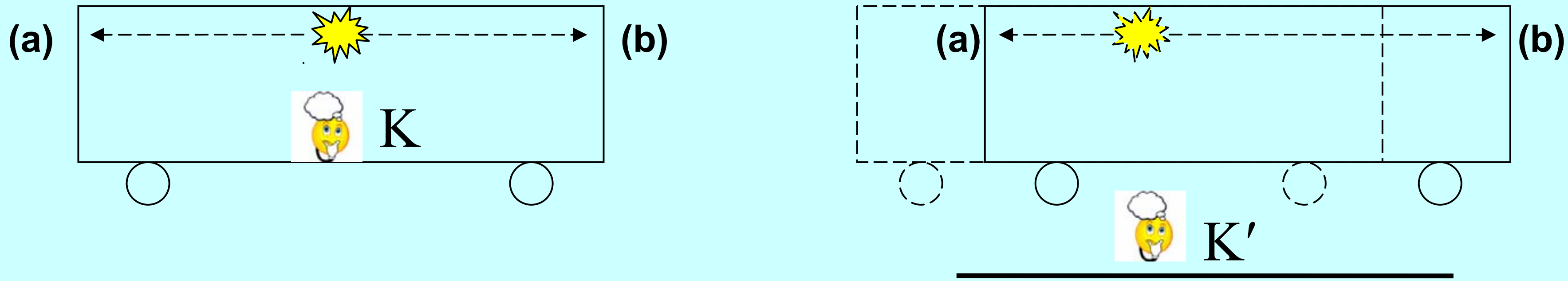
$$x' = x - Vt = x - \beta ct = 1x - 0.6ct = 0.4m$$
$$y' = y = 1m$$
$$z' = z = 1m$$
$$ct' = ct = 1m$$
$$(\text{thus } t' = 3.33\times10^{-9}s)$$

**Lorentz Tr.**

$$x' = x - \gamma\beta ct = 1.25x - 0.75ct = 0.5m$$
$$y' = y = 1m$$
$$z' = z = 1m$$
$$ct' = \gamma ct - \gamma\beta x = 1.25ct - 0.75x = 0.5m$$
$$(\text{thus } t' = 1.67\times10^{-9}s)$$

# *Relativity of simultaneity*

One of the striking consequences of Einstein's postulates in ***the relativity of simultaneity***.




***K*** – the rest reference frame of the car. For an observer in this IRF, light from the bulb in the middle of the car reaches the car ends (events (a) and (b)) ***simultaneously***.

***Second postulate***: the speed of light is the same in all IRFs!

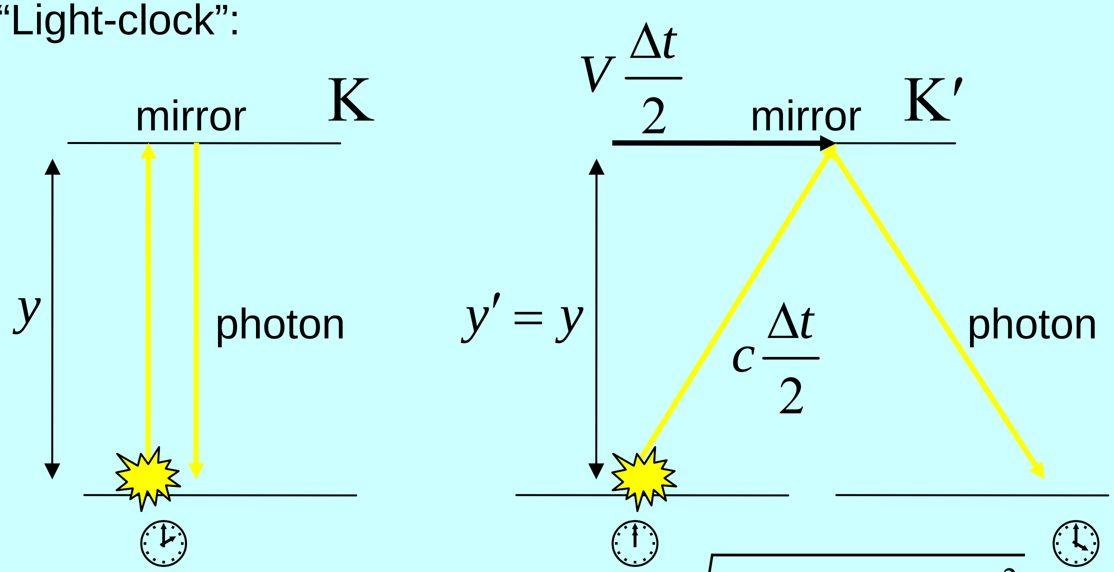
***K'*** – the ground reference frame. For this observer these two events are ***not*** simultaneous: as the light travels from the bulb, the train itself moves forward, and thus, event (a) happens ***before*** event (b).

***Two events that are simultaneous in one IRF are not, in general, simultaneous in another.***

# Time dilatation

**The proper time (interval), $\Delta t_0$ -** the time interval between two events occurring at the ***same position*** as measured by a clock at rest (with respect to these two events).

"Light-clock":




$$\Delta t_0 = \frac{2y}{c}$$

$$\Delta t = 2\frac{\sqrt{y^2 + (V\Delta t/2)^2}}{c}$$

***Second postulate***: the speed of light is the same in all IRFs!

We want to know the time interval between the same two events occurring at the same position as measured by (synchronized) clocks in a moving reference frame:

$$\Delta t = 2\frac{\sqrt{\left(c\frac{\Delta t_0}{2}\right)^2 + \left(V\frac{\Delta t}{2}\right)^2}}{c}$$

$$(c\Delta t)^2 = (c\Delta t_0)^2 + (V\Delta t)^2$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{V^2}{c^2}}} = \gamma \Delta t_0$$

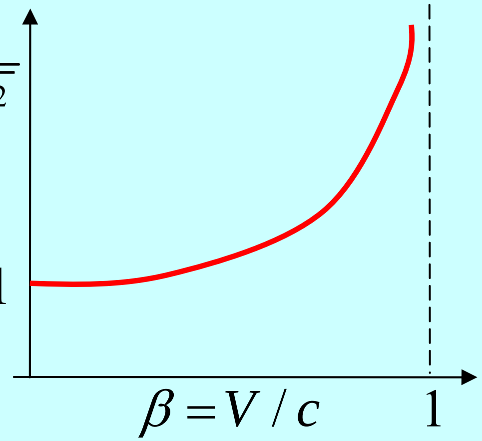
***The time interval measured in the moving system K′ is greater than the time interval measured in system K where these two events occur at the same place (the proper time is the minimum time interval).***

# Time dilatation

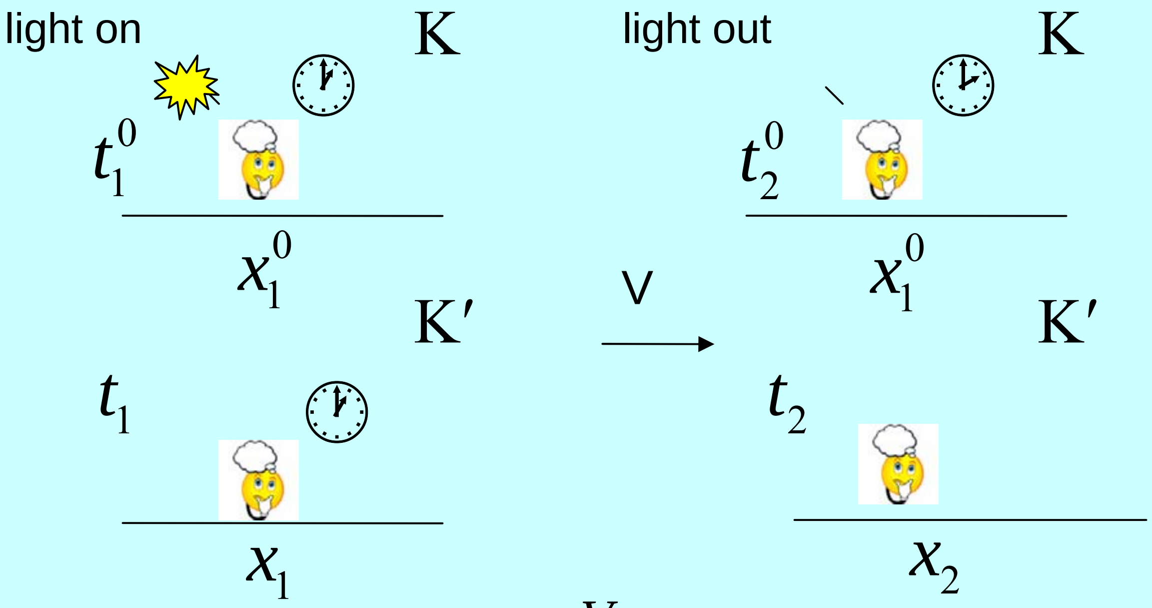
To observe this effect, the relative speed of the reference frames should be large. For the fastest spacecraft, the speed is ~$10^{-4}$c, and the effect is of an order of $10^{-8}$:

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\beta^2}} \approx \Delta t\left(1+\frac{\beta^2}{2}\right)$$

$$\frac{\Delta t}{\Delta t_0} = \frac{1}{\sqrt{1-\beta^2}}$$




Obviously, the same results stems directly from Lorentz transformations:




**Proper time interval:**

$$\Delta t_0 = t_2^0 - t_1^0$$

$$x_2^0 - x_1^0 = 0$$

$$t_2 - t_1 = \frac{\left(t_2^0 - t_1^0\right) - \frac{V}{c^2}\left(x_2^0 - x_1^0\right)}{\sqrt{1-V^2/c^2}}$$

but $x_2^0 = x_1^0$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-V^2/c^2}}$$

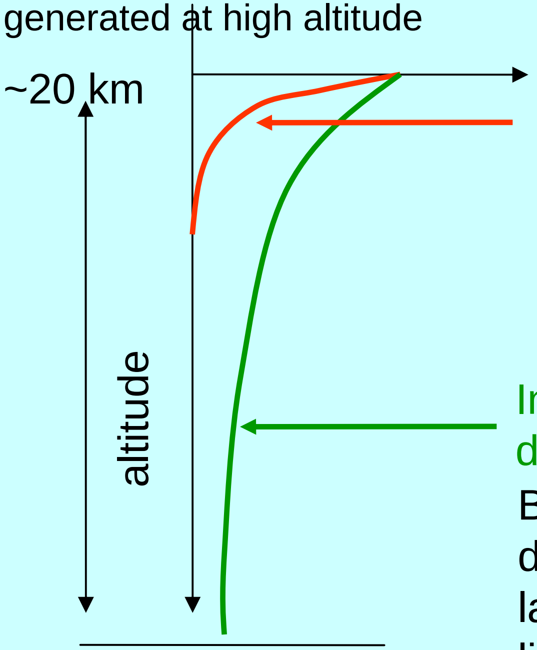
# *Consequences of the relativity principle*

- ***an event is completely defined by a space-time coordinate set***
- ***Lorentz transformations between two frames S & S' tells how a moving observer (S') would observe and event described by an observer (S) in the rest frame of an event***
- ***moving clocks run slower***

*Muons are created at high altitudes due to collisions of fast cosmic-ray particles (mostly protons) with atoms in the Earth atmosphere.*

*Muon – an electrically charged* ***unstable*** *elementary particle with a rest energy ~ 207 times greater than the rest energy of an electron. The muon has an average half-life of* $\mathbf{2.2 \times 10^{-6}}$ **s**.

$N_0$ – the number of muons generated at high altitude




$N$ – the number of muons measured in the sea-level lab

$$v = 2.994\times10^{8}\, m/s = 0.998c \qquad \beta = 0.998$$

In the muon's rest frame $\Delta t_0 = 2.2\times10^{-6}\, s$

By ignoring relativistic effects (wrong!), we get the decay length:

$$L = \Delta t_0 \times c = 2.2\times10^{-6}\, s \times 3\times10^{8}\, m/s = 660m$$

$$N = N_0 \exp\left(-\frac{20,000}{660}\right) = N_0 \exp(-30)$$

In fact, the decay length is much greater, the muons can be detected even at the sea level!

Because of the time dilation, in the RF of the lab observer the muon's lifetime is: $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-\beta^2}} \approx 35\times10^{-6}\, s$

$$L = 35\times10^{-6}\, s \times 3\times10^{8}\, m/s = 10.5km$$

$$N = N_0 \exp\left(-\frac{20,000}{10,500}\right) \approx N_0 \exp(-2)$$

# *Consequences of the relativity principle*

- *moving rods shrink in length along the line of motion*

- *quantity* **$[(c\cdot time)^2 - radius^2]$** **is the same in both frames**

- *physical laws are the same in any inertial frame*

- *Only requirements for S and S':*

  *S and S' should be in uniform relative motion*

# 4 - Vector

***A convenient and consistent method to denote an event***

- ***contravariant location 4-vector*** $x^\mu$ ***(***$\mu$ ***= 0,1,2,3)***

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) = (x^0, \vec{x})$$

- ***frequently used notation:*** $\tilde{x} \equiv (x^0, \vec{x})$

- ***covariant location 4-vector*** $x_\mu$ ***(***$\mu$ ***= 0,1,2,3)***

$$x_\mu = (x_0, x_1, x_2, x_3) \equiv (ct, -x, -y, -z)$$

- ***dot product:*** $x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (ct)^2 - x^2 - y^2 - z^2$

$$\Rightarrow \quad \tilde{x}\tilde{x} = (ct)^2 - x^2 - y^2 - z^2$$

- ***the scalar:*** $\tilde{a}\tilde{b}$ ***is invariant for all inertial frames***

# 4 - Vector

- ***corespondence between contra- and covariant 4-vectors***

$x_\mu = g_{\mu\nu} x^\nu$ *- summation convention*

*where* $$g^{\mu\nu} = g_{\mu\nu} \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ *metric tensor*

*some authors use different metric* $$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \Rightarrow$$ *different sign for the dot product*

*Lorentz transformation in 4-vector notation:*
$$\begin{cases} (x^0)' = \gamma(x^0 - \frac{\vec{\beta}}{c}\vec{x}) \\ (\vec{x})' = \vec{x} + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\vec{x})\vec{\beta} - \gamma x^0 \vec{\beta} \end{cases}$$

# 4 - Vector

• *inverse Lorentz transformation:*

  ∗ *exchange the primes ( ' )*

  ∗ *replace $\vec{\beta}$ by -$\vec{\beta}$*

• *some definitions using the metric $g^{\mu\nu}$ :*

$$x^\mu x_\mu > 0 \qquad \tilde{x} \quad \textit{is called } \textbf{timelike}$$

$$x^\mu x_\mu < 0 \qquad \tilde{x} \quad \textit{is called } \textbf{spacelike}$$

$$x^\mu x_\mu = 0 \qquad \tilde{x} \quad \textit{is called } \textbf{lightlike}$$

• *Lorentz transformation can be written in matrix notation:*

$$\begin{pmatrix} (x^0)' \\ (x^1)' \\ (x^2)' \\ (x^3)' \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

∗ *Easy to include rotations in the transformation*

# 4 - Vector

- *frequently used notations:* $\vec{x}^2 \equiv \vec{x} \cdot \vec{x}$ $\tilde{x}^2 \equiv \tilde{x} \cdot \tilde{x}$

- *Choise of* ***O = O'*** *at* ***t = t' = 0*** *was very special*

  $\rightarrow$ *allows comparison of absolute coordinates* ***x, x', …***

- *In general one measures differences in time and location*

  $*$ *general form of the invariant quantity:*

$$(c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \equiv (\Delta s)^2$$

*$\Delta$s = invariant “4-dimentional separation” of two events*

- *In most general differential notation:*

$$(ds)^2 = (cdt)^2 - (d\vec{r})^2 \quad \& \quad \Delta s = \int ds$$

*$(ds)^2$ defines the metric of space-time and thus all properties*

*(including the Lorentz transformations)*

# 4 - Vector

- *Let's define:* $(d\tau)^2 \equiv (ds)^2/c^2 \quad \& \quad \Delta\tau = \int d\tau$

*Timelike separation* $\rightarrow d\tau$ *= invariant time difference*

$\Delta\tau$ *is the time difference measured by a comoving clock*

*This time is called the* ***proper time***

- *Particle moving with velocity* $\vec{\mathbf{u}}$ *in the inertial frame* ***S***

*old fashion:* $\vec{u} = \frac{d\vec{r}}{dt} \quad \rightarrow \quad \textit{first guess} \quad \tilde{u} = \frac{d\tilde{x}}{dt}$

- ***d*** $\tilde{\mathbf{x}}$ *transforms as a 4-vecor but also* ***dt*** *transforms* $\Longrightarrow$ *this* $\tilde{\mathbf{u}}$ *does not transform as a 4 vector*

*A invariant time is necessary* $\rightarrow$ *use* $d\tau$

*definition of 4-velocity* $\tilde{u} = \frac{d\tilde{x}}{d\tau}$

$$t = \gamma_u \tau \quad \rightarrow \quad u^\mu = \gamma_u \frac{dx^\mu}{dt} \quad \rightarrow \quad \tilde{u} = \gamma_u(c, \vec{u})$$

*The norm of* $\tilde{\mathbf{u}}$ *is invarint* $\tilde{u}^2 = \gamma_u^2(c^2 - \vec{u}^2) = c^2$

# 4 - Vector

- *Leads to new expression for addition of velocities*

*- Frame **S'** moves with uniform $v\hat{x}$ relative to frame **S***

*- Particle moves with velocity $u'\hat{x}'$ relative to **S'** ($\hat{x}||\hat{x}'$)*

*Inverse Lorentz transformation gives **u***

$$\begin{cases} \gamma_u c = \gamma_v(\gamma_{u'} c + \beta_v \gamma_{u'} u') \\ \gamma_u u = \gamma_{u'} u' + \frac{\gamma_v - 1}{\beta^2}\beta^2 \gamma_{u'} u' - \gamma_c \gamma_{u'} c \beta_v = \gamma_v \gamma_{u'}(u' + v) \end{cases} \rightarrow \boxed{u = \frac{u' + v}{1 + \frac{u'v}{c^2}}}$$

$$\boxed{if\ u' = c \ \rightarrow \ u = c;\ c\ is\ really\ invariant\ !}$$

# 4 - Momentum

$$old\ fashion\ \vec{p} = m\vec{u}$$

$$definition\ of\ 4-momentum: \quad \tilde{p} \equiv m\tilde{u} \quad \tilde{u} = m\gamma_u(c, \vec{u})$$

- *Is this **4 – momentum** invariant ?*

$$\tilde{p}^2 = m^2\tilde{u}^2 = m^2\gamma_u^2(c^2 - u^2) = m^2c^2$$

- *What mean its components ?*

$$\tilde{p} = (p^0, \vec{p}) \quad \rightarrow \quad p^0 = m\gamma_u c \quad \& \quad \vec{p} = \gamma_u m\vec{u}$$

∗ ***New relativistic definition of $\vec{p}$ : extra factor $\gamma_u$***

- *What is the physical meaning of $p^0$ ?*

$$p^0 = mc\gamma_u = mc(1-\frac{u^2}{c^2})^{-1/2} = mc(1+\frac{1}{2}\frac{u^2}{c^2}+...) \rightarrow \quad p^0c = mc^2+\frac{1}{2}mu^2+...$$

***It's an energy!***

# 4 - Momentum

$$if\ \vec{u}=0 \ \rightarrow \ E_{u=0}=mc^2$$

*(rest energy)*

- *Definitions:*

*Total Energy :* ***E***

*Rest energy :* ***$E_0$***

*Kinetic energy :* ***T***

$$E=\gamma_u mc^2 \qquad \tilde{p}=(\frac{E}{c},\vec{p}) \qquad \tilde{p}^2=\frac{E^2}{c^2}-\vec{p}^2=m^2c^2$$

$$E^2=(\vec{p}c)^2+(mc^2)^2 \qquad E_0=mc^2 \qquad T=E-E_0$$

$$Basic\ 4\!-\!vectors \qquad \tilde{x},\tilde{u},\tilde{p} \quad \rightarrow \quad construct\ relativistic\ kinematics$$

$$4-Force \qquad (K^{\mu}) \quad \& \quad 4-Derivatives\ (\partial_{\mu})$$

$$K^{\mu}\equiv\frac{dp^{\mu}}{d\tau} \quad \partial_{\mu}\equiv\frac{\partial}{\partial x^{\mu}} \quad \partial_{\mu}=(\frac{\partial}{\partial x^0},\vec{\nabla})$$

*$\rightarrow$ **allow to develop Relativistic Dynamics***

# *Minkowski metric; rapidity*

- ***contravariant location 4-vector $x^\mu$ ($\mu$ = 0,1,2,3)***

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z) = (x^0, \vec{x})$$

- ***covariant location 4-vector $x_\mu$ ($\mu$ = 0,1,2,3)***

$$x_\mu = (x_0, x_1, x_2, x_3) \equiv (ct, -x, -y, -z)$$

$$x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (ct)^2 - x^2 - y^2 - z^2$$

- ***an alternative description of Lorentz transformation is based on the Minkowski trick of introducing an imaginary time $x_4 = it$***

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = it$$

- ***Lorentz invariant becomes:*** $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \text{invariant}$

- *The transformation which leaves this expression invariant is a rotation. A rotation in $(x_1, x_4)$ plane is:* $x'_1 = x_1 \cos\alpha - x_4 \sin\alpha, \qquad x'_4 = x_1 \sin\alpha + x_4 \cos\alpha$

- *In a real world these can be written:* $x' = x\cos\alpha - it\sin\alpha, \qquad it' = x\sin\alpha + it\cos\alpha$

# *Minkowski metric; rapidity*

*using the identities:* $i\sin\alpha = \sinh i\alpha, \qquad \cos\alpha = \cosh i\alpha$

*Taking **y=iα** the real form of Lorentz transformation is recovered:*

$$x' = x\cosh y - t\sinh y, \qquad t' = -x\sinh y + t\cosh y$$

*To be equivalent with:*

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{\beta}{c}x) \end{cases}$$

$\Longrightarrow$ *y must be real and α imaginary*

$\rightarrow$ *the price paid for a familiar Euclidean form of the rotation in ($x_1$,$x_4$) plane is an imaginary rotation angle*

*The real quantity y defined above is called **rapidity** of the transformation:*

$$\cosh y = \gamma, \qquad \sinh y = v\gamma \qquad\qquad \boxed{v = \tanh y}$$

*In ultrarelativistic regime, when particle mass can be neglected, i.e. **E=p**:* $y_{ur} = \ln(E'/E)$

# *Collisions' kinematics*

***The de Broglie relationship between wavelength and momentum p is:***

$$\lambda = h/p$$

$\Rightarrow$ $\lambda = 2\pi\hbar c/pc = 2\pi \times 197$ MeV fm/pc

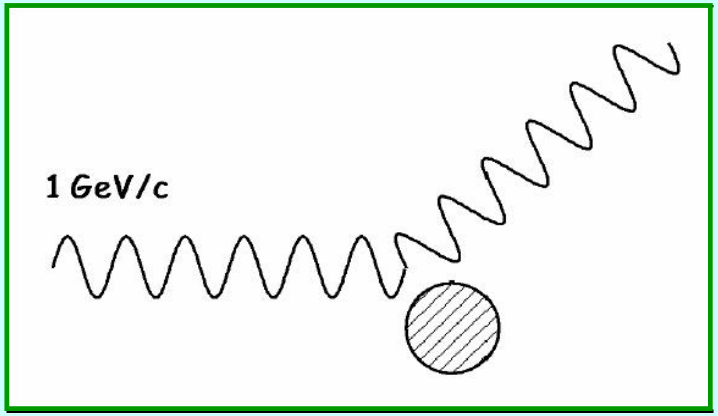
***where p is in MeV/c***

***For an electron with p = 1 GeV/c***

$\lambda = 2\pi \times 197/10^3 = 1.2$ fm or $1.2 \times 10^{-15}$ m

***Since a proton is ~1 fm in size,***

***a 1 GeV electron will see the whole proton***

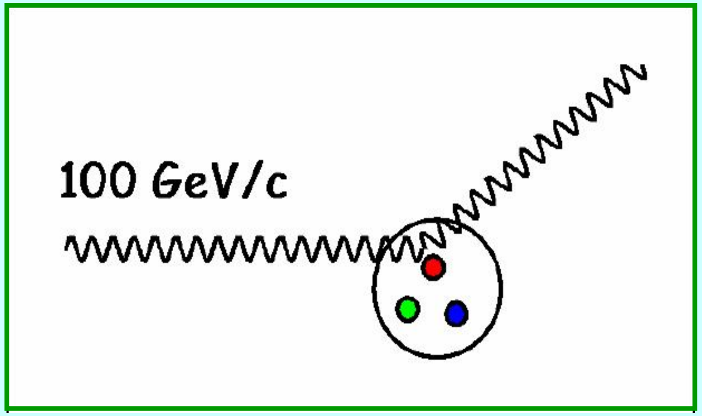


# *Collisions' kinematics*

***For an electron with p = 100 GeV-c:***

$\lambda = 2\pi \times 197/10^5 = 0.012$ ***fm or*** $1.2 \times 10^{-17}$***m***

⇒ ***a 100 GeV electron will be able to probe inside the proton and scatter off the quarks inside***




⇒ ***Increasing Energy*** ***Decreasing the probe size***

# *Collisions' kinematics*

- *Consider only pointlike particles*

- *Take into account all external effects*

  *- Friction → heat production*

  *radiated off → particles (photons) moving away*

*1. Within one reference frame* $\tilde{P}_{total}$ *is conserved (energy and momentum conservation)*

*2. Among reference frames in uniform relative motion* $\tilde{P}^2_{total}$ *is invariant*

$$E = \gamma_v mc^2 \quad \rightarrow \quad \gamma_v = \frac{E}{mc^2} \quad \rightarrow \quad \beta_v = \sqrt{1 - (\frac{mc^2}{E})^2}$$

- *Let's consider two particles, each with mass* ***m*** *and velocity* $v = 3/5c$

  *head-on collision & stick together*

$*$ *before collision :* $E_1 = E_2 \equiv E \quad \& \quad \vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$

$$\tilde{p}_1 = (\frac{E}{c}, \vec{p}) \quad \tilde{p}_2 = (\frac{E}{c}, -\vec{p}) \quad \rightarrow \quad \tilde{P}_{total} = (\frac{2E}{c}, 0)$$ $\Rightarrow$ ***The final partice is at rest***

# *Collisions' kinematics*

∗ *after collision :* $\tilde{P}'_{total} = (Mc, 0)$

$$\rightarrow \quad M = \frac{2E}{c^2} = 2\gamma_v m \quad \rightarrow \quad M = \frac{5}{2}m \quad (M > 2m)$$

∗ *Kinetic energy converted into rest energy !*

∗ *Classical:* ***E**$_{kin}$* $\leftrightarrow$ ***"internal energy"*** *(potential, heat, …)*

∗ *Relativistic: All **"internal "** forms of energy are reflected in the rest energy of the complete system*

• *Let's consider a proton and neutron at rest forming a deuteron with binding energy **B***

∗ *proton-neutron system :* $\tilde{P}_{total} = (m_p c + m_n c - \frac{B}{c}, \vec{0})$

∗ *deuteron :* $\tilde{P}_{total} = (Mc, \vec{0})$

$$M = m_p + m_n - \frac{B}{c^2}$$

# Collisions' kinematics

- Remaks :

1. **B** reflects the $\mathbf{E_{pot}}$ in the proton-neutron system
2. Inside the **p – n** system **B** is a separate component
3. Deuteron system as a whole → **B** reflected in rest energy
4. At least energy **B** needed to break-up the deuteron
   → the deuteron is a stable system

- In a laboratory **E** and $\vec{\mathbf{p}}$ of a particle are measured

$$E = \gamma_v mc^2 \quad \& \quad \vec{p} = \gamma_v m\vec{v} \quad \rightarrow \quad \vec{v} = \frac{\vec{p}c^2}{E}$$

# *Two body decay*

- *A pion ($\pi$) which is at rest decays into muon ($\mu$) and a massless neutrino ($\nu$)*

$*$ *Simple solution :*

- *Before decay :* $\tilde{P}_{total} = (m_\pi c, \vec{0})$

- *After decay :* $\tilde{p}_\mu = (\frac{E_\mu}{c}, \vec{p}_\mu) \quad \& \quad \tilde{p}_\nu = (\frac{E_\nu}{c}, \vec{p}_\nu) \quad \rightarrow \quad \tilde{P}'_{total} = (\frac{E_\mu + E_\nu}{c}, \vec{p}_\mu + \vec{p}_\nu)$

$$\vec{p}_\mu = -\vec{p}_\nu \equiv \vec{p} \quad m_\pi c^2 = E_\mu + E_\nu \quad E_\mu^2 = (m_\pi c^2 - E_\nu)^2$$

$$\Rightarrow \quad (\vec{p}c)^2 + (m_\mu c^2)^2 = (m_\pi c^2 - |\vec{p}|c)^2; \quad (m_\mu c^2)^2 = m_\pi^2 c^4 - 2m_\pi |\vec{p}| c^3$$

$$\rightarrow \quad |\vec{p}| = \frac{(m_\pi^2 - m_\mu^2)c}{2m_\pi}$$

# Two body decay

*Since:*

$$E_\mu^2 = (\vec{p}c)^2 + (m_\mu c^2)^2 \quad \rightarrow \quad E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}$$

$$\vec{v} = \frac{\vec{p}c^2}{E} \quad \rightarrow \quad v_\mu = \frac{(m_\pi^2 - m_\mu^2)c}{m_\pi^2 + m_\mu^2}$$

*∗ Let's use invariants :*

$$\tilde{p}_\pi = \tilde{p}_\mu + \tilde{p}_\nu \quad \rightarrow \quad \tilde{p}_\nu = \tilde{p}_\pi - \tilde{p}_\mu$$

$$\tilde{p}_\nu^2 = \tilde{p}_\pi^2 + \tilde{p}_\mu^2 - 2\tilde{p}_\pi \tilde{p}_\mu$$

$$0 = m_\pi^2 c^2 + m_\mu^2 c^2 - 2m_\pi E_\mu$$

$$\Rightarrow \quad E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}$$

$$E_\mu = \gamma_v m_\mu c^2 \quad \Rightarrow \quad \gamma_v = \frac{m_\pi^2 + m_\mu^2}{2m_\pi m_\mu} \quad \rightarrow \quad v = \frac{(m_\pi^2 - m_\mu^2)c}{m_\pi^2 + m_\mu^2}$$

# Two body decay of unstable particle in flight

*(c=1) !!!*

*mother particle* $\boldsymbol{P = (E,0,0,p)}$,

*daughter particles* $\boldsymbol{p_1 = (E_1, p_{1\perp}, p_{1z})}$ *and* $\boldsymbol{p_2 = (E_2, p_{2\perp}, p_{2z})}$

*due to momentum conservation* $\vec{p}_\perp \equiv \vec{p}_{1\perp} = -\vec{p}_{2\perp}$

*let's label the kinematical variables in CMS with asterisks*

$$\begin{aligned} E_1 &= \gamma(E_1^* + v p_{1z}^*) \\ p_{1z} &= \gamma(p_{1z}^* + v E_1^*) \\ \vec{p}_{1\perp} &= \vec{p}_{1\perp}^{\,*} \end{aligned}$$

*and similarly for particle 2*

$v = p/E$ *and* $\gamma = E/M$

*the problem is solved.* $\Longrightarrow$ *one could find now the angles which the two daughter particles make with the* $z$ *axis and with each other as function of the mother particle*

*(**Exercise 1**)*

*Lets solve the problem starting from energy-momentum conservation*

$$\begin{aligned} E &= E_1 + E_2 = \sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2} \\ \vec{p} &= \vec{p}_1 + \vec{p}_2 \end{aligned}$$

$$p_1 = \frac{(M^2 + m_1^2 - m_2^2)p\cos\theta_1 \pm 2E\sqrt{M^2 p^{*2} - m_1^2 p^2 \sin^2\theta_1}}{2(M^2 + p^2\sin^2\theta_1)}$$

# *Two body decay of unstable particle in flight*

*Reality of* $p_1 \rightarrow (M^2p^{*2} - m_1^2p^2\sin^2\theta_1) \geq 0$

- *condition satisfied for all* $\theta_1$ *if* $Mp^*/m_1p > 1$

  $\rightarrow$ *lower sign must be rejected since otherwise* $p_1$ *gets unphysical negative values for* $\theta_1 > \pi/2$

- *if* $Mp^*/m_1p < 1 \rightarrow$ *there is a maximum value of* $\theta_1$

  $\sin\theta_{1\,max} = Mp^*/m_1p \rightarrow$ *both signs must be kept:*

  *for each value of* $\theta_1 < \theta_{1\,max}$ *there are two values of* $p_1$ *and correspondingly for* $p_2$

***Exercise 2****: derive* $p_1$ *expression from the previous page*

***Exercise 3****: for the following decay processes:*

i. $K^o_S \rightarrow \pi^+ \pi^-$

ii. $\Lambda^0 \rightarrow p\pi^-$

iii. $\pi^0 \rightarrow 2\gamma$

*assuming 1 GeV LAB energy for the mother particle in each case:*

- *find the maximum Lab angles* $\theta$ *and corresponding momenta for all decay products*
- *show that the minimum opening angle between the decay products correspond to a CMS angle of* $90^0$ *with the flight direction of the mother particle and calculate the corresponding LAB momenta*
- *if a daughter particle makes half of maximum angle, find the corresponding two momenta and two momenta and LAB angles of other daughter particle*

# *Three-body decays; Dalitz plot*

*Consider the decay of mother particle M into three particles $m_1$, $m_2$ and $m_3$*

*Their corresponding 4-momenta are P $p_1$, $p_2$ and $p_3$*

*Energy-momentum conservation* $\rightarrow$ $\mathrm{P} = \mathrm{p}_1 + \mathrm{p}_2 + \mathrm{p}_3$

*Let's define the following invariants:*

$$\begin{aligned} s &= \mathrm{P}^2 = M^2 \\ s_1 &= (\mathrm{P} - \mathrm{p}_1)^2 = (\mathrm{p}_2 + \mathrm{p}_3)^2 \\ s_2 &= (\mathrm{P} - \mathrm{p}_2)^2 = (\mathrm{p}_3 + \mathrm{p}_1)^2 \\ s_3 &= (\mathrm{P} - \mathrm{p}_3)^2 = (\mathrm{p}_1 + \mathrm{p}_2)^2 \end{aligned}$$

*Which is their physical significance ?:*

- $s_1 = (p_2 + p_3)^2 \rightarrow (s_1)^{1/2}$ *is the invariant mass of the subsystem of particles 2 and 3*
- *Similarly* $(s_2)^{1/2}$ *and* $(s_3)^{1/2}$ *are the invariant masses of subsystems (3,1) and (1,2)*

*They are not independent:* $s_1 + s_2 + s_3 = M^2 + m_1^2 + m_2^2 + m_3^2$

# *Kinematical limits*

*In case of three particle decay arises the question of the limits of kinematical variables*

***phase space*** *– the space spanned by any set of independent kinematical variables*

*- with this definition one could say that we are deriving the boundaries of phase space*

*Let's consider the decay process in the rest frame of the mother particle (CMS):*

$$s_1 = M^2 + m_1^2 - 2ME_1$$

*with $E_1 = (m_1^2 + p_1^2)^{1/2}$ , $p_1$ is the CMS momentum of particle 1, $\rightarrow E_1 \geq m_1$, therefore:*

$$\max s_1 = (M - m_1)^2$$

*In order to find min $s_1$ , it has to be evaluated in the rest frame of subsystem (2,3) – Jackson frame*

*The kinematical variables in this frame are denoted by a little zero above de symbol*

$$s_1 = (\mathrm{p}_2 + \mathrm{p}_3)^2 = (\mathring{E}_2 + \mathring{E}_3)^2 \geq (m_2 + m_3)^2$$

$$\Rightarrow \qquad \begin{aligned} s_1 &\in \left[(m_2 + m_3)^2, (M - m_1)^2\right] \\ s_2 &\in \left[(m_3 + m_1)^2, (M - m_2)^2\right] \\ s_3 &\in \left[(m_1 + m_2)^2, (M - m_3)^2\right] \end{aligned}$$

# Kinematical limits

$$s_1 = (\mathring{E} - \mathring{E}_1)^2 = \left(\sqrt{M^2 + \mathring{p}_1^2} - \sqrt{m_1^2 + \mathring{p}_1^2}\right)^2$$

$$\mathring{p}_1^2 = \frac{1}{4s_1}\left[s_1 - (M - m_1)^2\right]\left[s_1 - (M + m_1)^2\right] \equiv \frac{1}{4s_1}\lambda\left(s_1, M^2, m_1^2\right)$$

$$s_1 = (\mathrm{p}_2 + \mathrm{p}_3)^2 = \left(\mathring{E}_2 + \mathring{E}_3\right)^2$$

$$\mathring{p}_2^2 = \mathring{p}_3^2 = \frac{1}{4s_1}\lambda\left(s_1, m_2^2, m_3^2\right)$$

*where* $\lambda(x,y,z)$ *Kinematical function is defined:* $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$

$$s_2 = (\mathrm{p}_1 + \mathrm{p}_3)^2 = m_1^2 + m_3^2 + 2\left(\mathring{E}_1\mathring{E}_3 - \mathring{p}_1\mathring{p}_3\cos\alpha\right)$$

*$\alpha$ Is the angle between $p_1$ and $p_3$*

$\Longrightarrow$ *$s_2$ depends on $\alpha$ if $s_1$ is fixed. It follows that $s_{2+} \equiv$ max $s_2$ and $s_{2-} \equiv$ min s2 Correspond to $\alpha = \pi$ and $\alpha = 0$, respectively, i.e.*

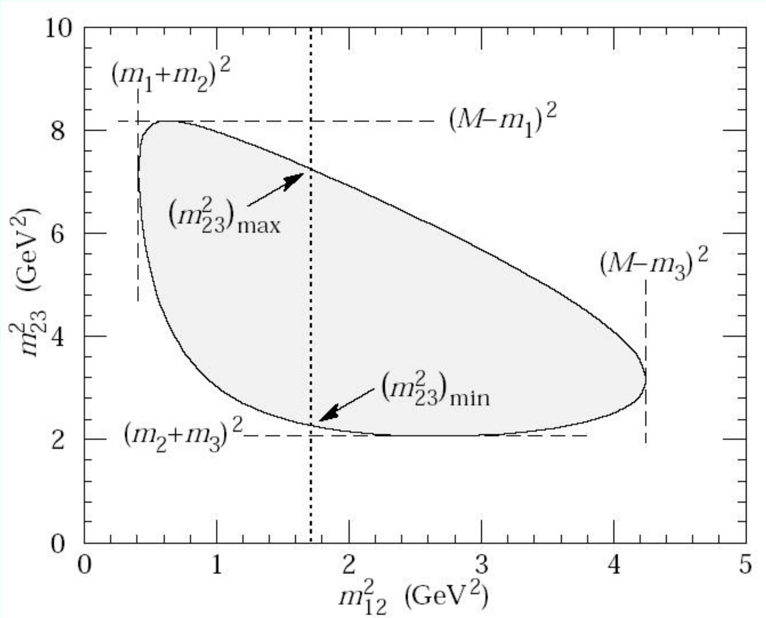
$$s_{2\pm} = m_1^2 + m_3^2 + 2\left(\mathring{E}_1\mathring{E}_3 \pm \mathring{p}_1\mathring{p}_3\right)$$

$$\mathring{E}_1 = \frac{1}{2\sqrt{s_1}}\left(s - s_1 - m_1^2\right), \qquad \mathring{E}_3 = \frac{1}{2\sqrt{s_1}}\left(s_1 + m_3^2 - m_2^2\right)$$

# Kinematical limits

$$s_{2\pm} = m_1^2 + m_3^2 + \frac{1}{2s_1}\left[\left(s - s_1 - m_1^2\right)\left(s_1 - m_2^2 + m_3^2\right) \pm \lambda^{1/2}(s_1, s, m_1^2)\lambda^{1/2}(s_1, m_2^2, m_3^2)\right]$$

*This equation defines a curve which is the boundary of the* ***Dalitz plot*** *in ($s_1$,$s_2$) plane*




*$\pi^+ \bar{K}^0 p$ at 3 GeV*

*The maximum values of the three-momenta of daughter particles in the rest frame of the mother particle:*

$$p_{1max} = \frac{1}{2M}\sqrt{[M^2 - (m_1 + m_2 + m_3)^2][M^2 - (m_2 + m_3 - m_1)^2]}$$

*Similar expressions are obtained for $p_{2max}$ and $p_{3max}$ by cyclic replacement of subscripts:*

$$1 \to 2,\ 2 \to 3,\ 3 \to 1$$

# Particle collisions
# (centre-of-mass frame and laboratory frame)

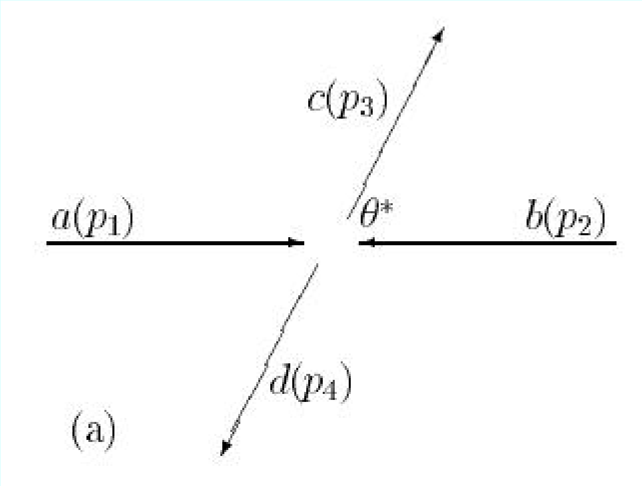
*The total 4-momentum of a system of n particles* $p_1^\mu,\ p_2^\mu,\ \ldots, p_n^\mu$

$$p^\mu = p_1^\mu + p_2^\mu + \ldots + p_n^\mu$$

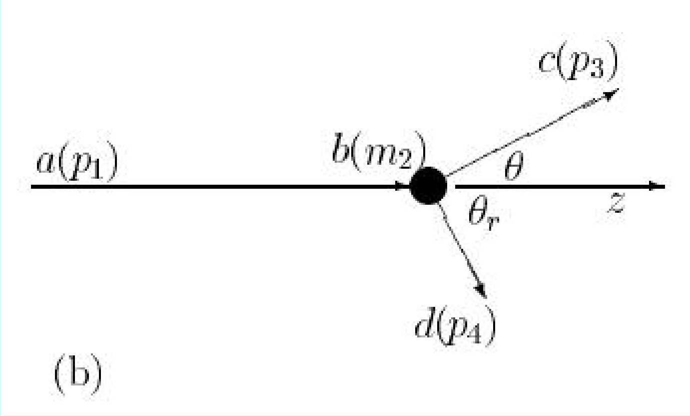
*Labeling CMS variables by asterisks, the CMS is defined as:*

$$\vec{p}^{\,*} = \vec{p}_1^{\,*} + \vec{p}_2^{\,*} + \ldots + \vec{p}_n^{\,*} = 0$$

*The LAB frame is defined as the reference frame in which one of the initial particles is at rest. This particle is called the target particle, the other being the beam particle or incident particle.*




CMS




LAB

# Particle collisions (centre-of-mass frame and laboratory frame)

$$p_1 = (E_{LAB}, 0, 0, p_{LAB})$$ *beam particle*

$$p_2 = (m_2, 0, 0, 0)$$ *target particle*

$$\Rightarrow \quad p = p_1 + p_2$$

*the invariant square of p is usually denoted by s*

$$s = (E_1^* + E_2^*)^2$$

$\Rightarrow$ *$s^{1/2}$ is the total CMS energy of the system*

$$E_1^* = \sqrt{m_1^2 + p^{*2}}, \qquad E_2^* = \sqrt{m_2^2 + p^{*2}}$$

$$p^* = \frac{1}{2\sqrt{s}} \left\{ \left[ s - (m_1 - m_2)^2 \right] \left[ s - (m_1 + m_2)^2 \right] \right\}^{1/2}$$

*if particles c and d are different from a and b, then:*

$$p_f^* = \frac{1}{2\sqrt{s}} \left\{ \left[ s - (m_3 - m_4)^2 \right] \left[ s - (m_3 + m_4)^2 \right] \right\}^{1/2}$$

*in the LAB frame s is given by*

$$s = p^\mu p_\mu = m_1^2 + m_2^2 + 2 m_2 E_{LAB}$$

$$\Rightarrow \quad E_{LAB} = (s - m_1^2 - m_2^2)/2m_2$$

# Particle collisions
# (centre-of-mass frame and laboratory frame)

$$E_{LAB} = \sqrt{p_{LAB}^2 + m_1^2}$$

$$\Rightarrow \quad p_{LAB} = \frac{1}{2m_2}\left\{\left[s - (m_1 - m_2)^2\right]\left[s - (m_1 + m_2)^2\right]\right\}^{1/2} = \frac{1}{2m_2}\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)$$

$$p^* = p_{LAB}\frac{m_2}{\sqrt{s}}$$

$$E_{1,2}^* = (m_{1,2}^2 + m_2 E_{LAB})/\sqrt{s}$$

*For ultra-relativistic particles, i.e. energies where one could neglect all masses in comparison with the particle energies, $E^*_1 = E^*_2$*

$$\Rightarrow \quad E^* \approx \sqrt{\frac{1}{2}m_2 E_{LAB}}$$

$\Rightarrow$ *In fixed target experiments at high energies most of the beam energy goes in the kinetic energy of the particle system $\rightarrow$ colliders, i.e. accelerators with two beams moving in opposite directions*

*Example: - in an e+e- collider one needs two beams of half Z boson mass, i.e. ~ 45.5 GeV each to produce it*

*- The equivalent LAB energy of a e+ on a target e- is 8.3 x $10^6$ GeV !!!*

# Particle collisions
# (centre-of-mass frame and laboratory frame)

*Let's calculate the LAB velocity of the centre of mass of the initial particles.*

$$E_1 + E_2 \equiv E_{LAB} + m_2 = \gamma_{cm}[E_1^* + E_2^* + v_{cm}(p_{1z}^* + p_{2z}^*)]$$

*using* $s = (E_1^* + E_2^*)^2$ *and* $p^*_{1z} + p^*_{2z} = 0$

$$\Rightarrow \quad \gamma_{cm} = (E_{LAB} + m_2)/\sqrt{s}$$

$$v_{cm} = p_{LAB}/(E_{LAB} + m_2)$$

*For equal mass particles - in the nonrelativistic limit the centre of mass moves with half the speed of incident particle*

*- the relativistic expression is:*

$$v_{cm} = \sqrt{\frac{E_{LAB} - m}{E_{LAB} + m}}$$

*For* $p_{LAB} \ll m$

$$v_{cm} = \frac{p_{LAB}}{2m}\left(1 - \frac{p_{LAB}^2}{4m^2}\right)$$

$v_{LAB} = p_{LAB}/\gamma_{LAB} m$ *and* $\gamma_{LAB} = (1 - v^2_{LAB})^{-1/2}$ $\rightarrow$

$$v_{cm} = \frac{1}{2} v_{LAB}\left(1 + \frac{1}{4} v_{LAB}^2\right)$$

*One can use* $v_{cm}$ *and* $\gamma_{cm}$ *to write down the Lorentz transformation from LAB to CMS*

$$E_1^* = \gamma_{cm}(E_{LAB} - v_{cm} p_{LAB}) = (m_1^2 + m_2 E_{LAB})/\sqrt{s}$$

$$E_2^* = \gamma_{cm} m_2 = m_2(m_2 + E_{LAB})/\sqrt{s}$$

# *Elastic collisions*

$$p_3^2 = p_1^2 = m_1^2 \quad \text{and} \quad p_4^2 = p_2^2 = m_2^2$$

$$p_1 + p_2 = p_3 + p_4$$

*One could construct sixteen $p_i \cdot p_j$, $i,j = 1, 2, 3, 4$ invariants*

*- four are $p_i^2 = m_i^2$ – trivial invariants, i.e. constants without any dynamical content*

*- as far as* $p_i \cdot p_j = p_j \cdot p_i$

$\Longrightarrow$ *Remain six invariants constrained by 4-mometum conservation*

$\Longrightarrow$ *Remain two independent variables*

*It is frequently convenient to use three invariants with one constraint.*

*The most used choice are the Mandelstam variables:*

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned}$$

*Since only two of them are independent there exists one relation between them:*

$$s + t + u = 2m_1^2 + 2m_2^2$$

# *Elastic collisions*

*t has a special meaning in the CMS where $E^*_1 = E^*_2$*

$$t = -(\vec{p}_1 - \vec{p}_3)^2 = -2p^{*2}(1 - \cos\theta^*)$$

*- up to a sign, t is the square of the momentum transfer in the CMS*

*- In the elastic scattering t is always negative except at $\theta^* = 0$ (forward scattering) where $t = 0$*

# *Inelastic collisions*

$$\pi^+ + p \to \pi^+ + p + \pi^+ + \pi^-$$

$$e^+ + e^- \to \mu^+ + \mu^-$$

$$e^- + p \to e^- + X$$

***inclusive collision*** *X – any system of final state particles*

*energy and momentum conservation* $\mathrm{p}_1 + \mathrm{p}_2 = \mathrm{p}_3 + \mathrm{p}_4 + \ldots + \mathrm{p}_n$

*Which is the threshold energy for this reaction?*

$$\mathrm{p}_1 = (E_{LAB}, 0, 0, p_{LAB}) \qquad \mathrm{p}_2 = (m_2, 0, 0, 0)$$

$$\Rightarrow \quad s = (\mathrm{p}_1 + \mathrm{p}_2)^2 = m_1^2 + m_2^2 + 2m_2 E_{LAB}$$

$$\Rightarrow \quad s = (\mathrm{p}_3 + \mathrm{p}_4 + \ldots + \mathrm{p}_n)^2$$

*CMS expression of s and using energy and momentum conservation*

$$s = (E_3^* + E_4^* + \ldots + E_n^*)^2 \geq (m_3 + m_4 + \ldots + m_n)^2$$

*where we used:* $E_i^* = \sqrt{m_i^2 + \vec{p}_i^{*2}} \geq m_i.$

# *Inelastic collisions*

$$E^*_{thr} = \sqrt{s_{min}} = m_3 + m_4 + \ldots + m_n$$

*setting* $M = m_3 + m_4 + \cdots + m_n$

$$E^{thr}_{LAB} = \frac{1}{2m_2}\left[M^2 - m_1^2 - m_2^2\right]$$

*introducing the LAB kinetic energy* $T_{LAB} = E_{LAB} - m_1$

$$T^{thr}_{LAB} = \left[M^2 - (m_1 + m_2)^2\right]/2m_2$$

*Example:*

*- the threshold kinetic energy for* $\pi^+ + p \rightarrow \pi^+ + p + \pi^+ + \pi^+$

*using the approximate mass values* $m_p$ = *940 MeV,* $m_\pi = m_{\pi+} = m_{\pi-}$ = *140 MeV*

$$\begin{aligned} T^{thr}_{LAB} &= \frac{1}{2m_p}\left[(m_p + 3m_\pi)^2 - (m_p + m_\pi)^2\right] \\ &= \frac{1}{2m_p}(2m_p + 4m_\pi)(2m_\pi) = 2m_\pi\left(1 + \frac{2m_\pi}{m_p}\right) \\ &= 2 \times 140 \times (1 + 2 \times 140/940) = 363.4 \text{ MeV} \end{aligned}$$

# *Quasi-elastic collisions*

*particular case of inelastic collisions, i.e. 2 → 2 body reactions*

*i.e.* $e^+ + e^- \rightarrow \mu^+ + \mu^-$

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

$$\vec{p}_1^* + \vec{p}_2^* = \vec{p}_3^* + \vec{p}_4^*$$

$$|\vec{p}_1^*| \neq |\vec{p}_3^*| \qquad p^* = |\vec{p}_1^*| = |\vec{p}_2^*| \quad \text{and} \quad p^{*\prime} = |\vec{p}_3^*| = |\vec{p}_4^*|$$

$$p^{*\prime} = \frac{1}{2\sqrt{s}} \left\{ \left[ s - (m_3 - m_4)^2 \right] \left[ s - (m_3 + m_4)^2 \right] \right\}^{1/2})$$

*Example:* $\pi^- + p \rightarrow K^- + \Sigma^+$

$m_K = 0.494$ GeV $\quad m_{\Sigma^+} = 1.189$ GeV $\Rightarrow$ *threshold energy is 1.03 GeV*

*Assuming the LAB energy of the incident pion of 1.5 GeV*

$$\Rightarrow \quad s = m_\pi^2 + m_p^2 + 2m_p E_{LAB} = 0.14^2 + 0.94^2 + 2 \times 0.94 \times 1.5 = 3.71 \text{ GeV}^2$$

$$p^* = 0.727 \text{ GeV} \quad \text{and} \quad p^{*\prime} = 0.438 \text{ GeV}$$

# Quasi-elastic collisions

***CMS variables:***

$$E_1^* = \frac{s+m_1^2-m_2^2}{2\sqrt{s}}, \qquad E_2^* = \frac{s+m_2^2-m_1^2}{2\sqrt{s}}$$

$$E_3^* = \frac{s+m_3^2-m_4^2}{2\sqrt{s}}, \qquad E_4^* = \frac{s+m_4^2-m_3^2}{2\sqrt{s}}$$

$$p_1^* = p_2^* = \frac{1}{2\sqrt{s}}\sqrt{\left[s-(m_1-m_2)^2\right]\left[s-(m_1+m_2)^2\right]}$$

$$p_3^* = p_4^* = \frac{1}{2\sqrt{s}}\sqrt{\left[s-(m_3-m_4)^2\right]\left[s-(m_3+m_4)^2\right]}.$$

*denoting the CMS angle between $\vec{p}_3{}^*$ and $\vec{p}_1{}^*$ by $\theta^*$*

$$\cos\theta^* = 1-(t_0-t)/2p_1^*p_3^* = 1-(u-u_0)/2p_1^*p_4^*$$

$$t_0 = m_1^2+m_3^2-2(E_1^*E_3^*-p_1^*p_3^*) = m_2^2+m_4^2-2(E_2^*E_4^*-p_2^*p_4^*)$$

$$u_0 = m_1^2+m_4^2-2(E_1^*E_4^*+p_1^*p_4^*) = m_2^2+m_3^2-2(E_2^*E_3^*+p_2^*p_3^*)$$

***LAB variables:***

$$E_1 = \frac{s-m_1^2-m_2^2}{2m_2}, \qquad E_2 = m_2$$

$$E_3 = \frac{m_2^2+m_3^2-u}{2m_2}, \qquad E_4 = \frac{m_2^2+m_4^2-t}{2m_2}$$

*The angle between $\vec{p}_3$ with z axis is given by:*

$$\cos\theta_3 = \frac{t-m_1^2-m_3^2+2E_1E_3}{2p_1p_3}$$

*and similarly*

$$\cos\theta_4 = \frac{u-m_1^2-m_4^2+2E_1E_4}{2p_1p_4}$$

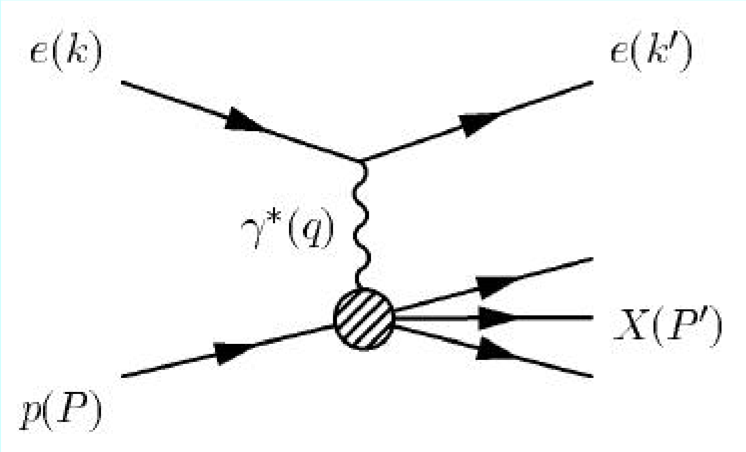
# Deep inelastic scattering

*In lepton-hadron scattering at high energies a large number of hadrons are produced In the final state: this is called **deep inelastic scattering (DIS)***

$$e^- + p \to e^- + X$$

*X stands for the hadronic system with an arbitrary number of particles*

*A generic diagram looks like:*




*To describe the DIS reaction kinematics we make the following notations:*

*- 4-momentum of the incoming electron k = (E,0,0,k)*

*- 4-momentum of the target proton by P*

*- 4-momentum of the scattered electron k'*

*- 4-momentum of the hadronic system P'*

*- 4-momentum of the exchanged virtual photon $\gamma^*$ q = k - k'*

# Deep inelastic scattering

*4-momentum conservation* → $\mathrm{k} + \mathrm{P} = \mathrm{k}' + \mathrm{P}'$

*And we have the mass shell conditions* $k^2 = k'^2 = m_e^2$ *and* $P^2 = m_p^2$

*Since the energies characteristic of DIS are at least several GeV,*

*the electron mass can be set equal to zero* → *the square of the 4-momentum transfer is*

$$q^2 = (k - k')^2 = -2EE'(1 - \cos\theta)$$

→ $q^2 \leq 0$, i.e. the exchange photon is space-like !

*The invariant* $W = P'^2$ *is variable because the multiplicity of particles in the hadronic system is variable*

→ *the complete kinematics of DIS is determined by* ***three independent invariants***

*relative*

*to* ***two*** *as we are used for inelastic collisions*

*- One is the square of the total CMS energy S:* $S = (\mathrm{k} + \mathrm{P})^2 = m_p^2 + 2\mathrm{k} \cdot \mathrm{P}$

*- The second invariant is usually chosen to be the negative square of the 4-momentum transfer:*

$$Q^2 = -\mathrm{q}^2 = -(\mathrm{k} - \mathrm{k}')^2 = 4EE' \sin^2 \frac{\theta}{2}$$

*-The third independent invariant can be taken to be W or*

*alternatively one of the dimensionless variables:*

$$x = \frac{Q^2}{2\mathrm{P} \cdot \mathrm{q}} \quad \text{or} \quad y = \frac{\mathrm{P} \cdot \mathrm{q}}{\mathrm{k} \cdot \mathrm{P}}$$

# Deep inelastic scattering

The variable $y$ has a simple physical meaning in the target frame where

$P = (m_p,0,0,0)$, $k = (E_{LAB},0,0,E_{LAB})$ and $k' = (E'_{LAB},\vec{p}_3)$

$\rightarrow$ $y = 1 - E'_{LAB}/E_{LAB}$, i.e. $y$ is the relative energy loss of the electron in the LAB

The invariant $x$ is Bjorken scaling variable or simply Bjorken-$x$,

Bjorken predicting the property of of scaling in DIS, confirmed experimentally

Interesting is the expression of S in terms of beam energies.

- fixed target DIS, e or $\mu$ beam with 4-momentum $k = (E,0,0,E)$ and p target with $P = (E_p,0,0,0)$

$$\Longrightarrow \quad S = m_p^2 + 2m_pE$$

- e – p collider like HERA the 4-momenta of the colliding particles are

$P = (E_p,0,0,E_p)$ and $k = (E_e,0,0,-E_e)$

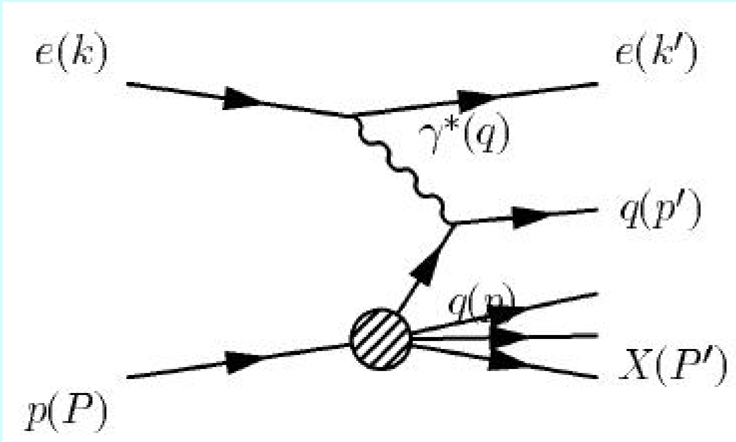
$$\Longrightarrow \quad S = 4E_eE_p$$

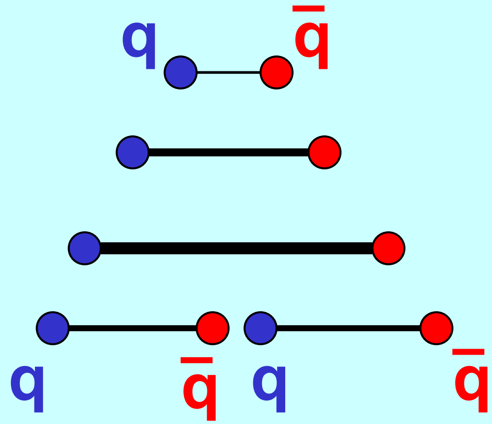
Other useful relations between various kinematical variables:

$$Q^2 = xyS \qquad \text{and} \qquad W^2 = m_p^2 + Q^2(1/x - 1)$$

# *Deep inelastic scattering*

*Within the framework of parton model, DIS proceeds by exchange of a photon or intermediate vector boson with only one of the quarks of the proton:*




*The electron – quark collision is elastic $\rightarrow$ the struck quark acquires a sufficient momentum to break away from the rest of proton as far as the color force allows it to travel*

$\rightarrow$ *at this stage some of the binding energy is converted in creation of a quark-antiquark pair from vacuum*

*- the anti-quark combines with the original quark into a meson, leaving behind an other quark which can give rise to the creation of another quark-antiquark pair*

$\rightarrow$ ***fragmentation*** *continues until the remaining energy drops bellow the pair creation threshold*

# *Deep inelastic scattering*

*The result - several mesons are created which travel roughly in the direction of struck quark*

*- such a system of mesons or more generally hadrons is called* ***jet***

*- the residue of proton is highly unstable system: - lost a quark*

*- absorbed a quark presumably of the wrong sort left over from the fragmentation*

*- has absorbed a fraction of the energy transferred from the electron*

*$\rightarrow$ it breaks up into several hadrons*

*- the elastic electron-quark collision is the* ***hard subprocess*** *of DIS*

*At high momentum, where the proton mass is negligible, the quark energy is the same fraction of proton energy.*

*It turns out that this fraction is identical with the Bjorken-x defined above*

*Denoting the 4-momentum of the incoming quark by p we have therefore: $p = xP$*

*Denoting the invariant $(k + p)^2$ by s, which is the squared CMS energy of the subprocess*

*we have also : $s = xS$*

*This shows that the two independent invariants that control the kinematics of the subprocess are $x$ and $Q^2$*

# Lorentz invariant phase space

***Fermi's golden rule***

$$W = \frac{2\pi}{\hbar} \left| m_{if} \right|^2 \rho(E)$$

*Transition rate*

*density of states available for energy E (phase space factor)*

*Transition matrix element*

$i \to f \quad |\langle \Psi_f | V_{if} | \Psi_i \rangle|^2$

- $|m_{if}|^2$ *may be unknown*
- *therefore, we need to calculate* $\rho(E)$ *to extract the dynamics of the matrix element*

***Density of states***

- *state of motion of a single particle with momentum between 0 and p confined to volume V is specified by 6-D phase space* $(x,y,z,p_x,p_y,p_z)$
- *momentum and space coordinate can be specified within h limit – from uncertainty principle* $\to$ *element volume of space phase is* $h^3$
- *therefore the number of states available to an individual particle,* $N_i$ *, is:*

$$N_i = \frac{\text{total phase space volume}}{\text{elementary volume}} = \frac{1}{(2\pi\hbar)^3} \int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = \frac{V}{(2\pi\hbar)^3} \int d^3 p$$

# *Lorentz invariant phase space*

• *For a system of n particles the number of available final states, $N_n$, is the product of the individual particles' number of states:*

$$N_n = \left( \frac{V}{(2\pi\hbar)^3} \right)^n \int \prod_{i=1}^{n} d^3 p_i$$

***Phase space***

• *Definition: the phase space factor is the number of states per unit of energy interval per unit volume (V=1) :*

$$\rho(E) = \frac{dN_n}{dE} = \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3 p_i$$

• *Due to momentum conservation not all momenta are independent:*

$$\left( \sum_{i=1}^{n} \mathbf{p}_i \right) - \mathbf{P} = 0$$

• *Can be accomodated by integrating over n-1 particles:*

$$\rho(E) = \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n-1} d^3 p_i$$

# *Lorentz invariant phase space*

•*The momentum conservation can be considered by using Dirac $\delta$ function:*

$$\mathbf{p_n} - \left(\mathbf{P} - \sum_{i=1}^{n-1} \mathbf{p}_i\right) = 0 \text{ so } \int d^3 p_n \delta\left[\mathbf{p_n} - \left(\mathbf{P} - \sum_{i=1}^{n-1} \mathbf{p}_i\right)\right] = 1$$

$$\rho(E) = \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n-1} d^3 p_i = \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3 p_i \delta\left[\mathbf{p_n} - \left(\mathbf{P} - \sum_{i=1}^{n-1} \mathbf{p}_i\right)\right]$$

$$= \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3 p_i \delta\left[\mathbf{P} - \sum_{i=1}^{n} \mathbf{p}_i\right]$$

•*Energy conservation gives:* $\sum_{i=1}^{n} E_i - E = 0$ $\quad \int dE \delta\left(\sum_{i=1}^{n} E_i - E\right) = 1$

$$\rho(E) = \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3 p_i dE \delta\left[\mathbf{P} - \sum_{i=1}^{n} \mathbf{p}_i\right] \delta\left(\sum_{i=1}^{n} E_i - E\right)$$

$$= \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3 p_i \delta\left[\mathbf{P} - \sum_{i=1}^{n} \mathbf{p}_i\right] \delta\left(\sum_{i=1}^{n} E_i - E\right) \text{ as } \frac{d}{dE} \int f(E) dE = f(E)$$

*The only problem: this* ***is not Lorentz invariant***

# Lorentz invariant phase space

***Insuring Lorentz invariance***

- *Fermi's golden rule:* $W = \frac{2\pi}{\hbar}\left|m_{if}\right|^2 \rho(E)$

- *If $\rho(E)$ is not Lorentz invariant $\rightarrow |m_{if}|^2$ is not*

- *a single massive particle moving with energy E in a volume V described by a wavefunction $\Psi$ normalized to $\int|\Psi|^2 dV = 1$*

- *This normalization implies that particle density is 1/V for a stationary observer*

- *if the particle speed is relativistic then there will be a contraction by a factor $1/\gamma$ in the direction of motion $\rightarrow$ the particle density appears to be $\gamma/V$*

- *renormalizing the wave functions to $\Psi' \rightarrow \gamma^{1/2}\Psi$ guaranties that the particle density becomes invariant*

- *we can redifine the matrix element:*

$$\left|M_{if}\right|^2 = \left|m_{if}\right|^2 \prod_{j=1}^{n} 2m_j\gamma_j c^2 \prod_{i=1}^{n} 2m_i\gamma_i c^2 = \left|m_{if}\right|^2 \prod_{j=1}^{n} 2E_j \prod_{i=1}^{n} 2E_i$$

# *Lorentz invariant phase space*

$\Longrightarrow$ *Transition rate to a single final state becomes:*

$$dW = \frac{2\pi}{\hbar}\frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j}\frac{1}{(2\pi\hbar)^{3n}}\left(\prod_{i=1}^{n}\frac{d^3 p_i}{2E_i}\delta\left(\sum_{i=1}^{n}\mathbf{p}_i - \mathbf{P}\right)\delta\left(\sum_{i=1}^{n}E_i - E\right)\right)$$

*Integrate over all final states to get:*

$$W = \frac{2\pi}{\hbar}\frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j}\frac{1}{(2\pi\hbar)^{3n}}\int\left(\prod_{i=1}^{n}\frac{d^3 p_i}{2E_i}\delta\left(\sum_{i=1}^{n}\mathbf{p}_i - \mathbf{P}\right)\delta\left(\sum_{i=1}^{n}E_i - E\right)\right) = \frac{2\pi}{\hbar}\frac{\left|M_{if}\right|^2}{\prod_{j=1}^{n} 2E_j}R_n(E)$$

## *Lorentz invariant phase space*

$$R_n(E) = \frac{1}{(2\pi\hbar)^{3n}}\int\prod_{i=1}^{n}\frac{d^3 p_i}{2E_i}\delta\left(\sum_{i=1}^{n}\mathbf{p}_i - \mathbf{P}\right)\delta\left(\sum_{i=1}^{n}E_i - E\right)$$

# *Lorentz invariant phase space*

***Is it invariant ?***

$$p'_x = p_x \quad p'_y = p_y \quad p'_z = \gamma\left(p_z - \beta E\right) \quad E' = \gamma\left(E - \beta p_z\right)$$

$$\frac{dp'_z}{dp_z} = \gamma\left(1 - \beta\frac{dE}{dp_z}\right) = \gamma\left(1 - \beta\frac{p_z}{E}\right)$$

$$\frac{dE}{dp_z} = \frac{d}{dp_z}\left(\sum_{i=xyz} p_i^2 + m^2\right)^{\frac{1}{2}} = p_z\left(\sum_{i=xyz} p_i^2 + m^2\right)^{-\frac{1}{2}} = \frac{p_z}{E}$$

$$\frac{dp'_z}{dp_z} = \gamma\left(1 - \beta\frac{p_z}{E}\right) = \frac{\gamma\left(E - \beta p_z\right)}{E} = \frac{E'}{E}$$

$$\Rightarrow \quad \frac{dp'_z}{E'} = \frac{dp_z}{E} \qquad \frac{d^3 p'}{E'} = \frac{d^3 p}{E}$$

## *2 body phase space*

$$R_2(E) = \frac{1}{(2\pi\hbar)^6} \int \prod_{i=1}^{2} \frac{d^3 p_i}{2E_i} \delta\left(\sum_{i=1}^{2} \mathbf{p}_i - \mathbf{P}\right) \delta\left(\sum_{i=1}^{2} E_i - E\right)$$

$$= \frac{1}{(2\pi\hbar)^6} \iint \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) \delta(E_1 + E_2 - E)$$

$$= \frac{1}{(2\pi\hbar)^6} \iint \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \delta(\mathbf{p}_1 + \mathbf{p}_2) \delta(E_1 + E_2 - E) \quad \text{in centre of mass frame}$$

$$= \frac{1}{(2\pi\hbar)^6} \int \frac{d^3 p_1}{4E_1 E_2} \delta(E_1 + E_2 - E) \quad \text{integrate over } \mathbf{p}_2$$

$$= \frac{1}{(2\pi\hbar)^6} \int \frac{4\pi p_1^2 dp_1}{4E_1 E_2} \delta(E_1 + E_2 - E)$$

$$= \frac{\pi}{(2\pi\hbar)^6} \int \frac{p_1 dE_1}{E_2} \delta(E_1 + E_2 - E) \quad \text{as } p_1 dp_1 = E_1 dE_1 \text{ from } E_1^2 - p_1^2 = m_1^2$$

# 2 body phase space

*In order to integrate we need to write $E_2$ in terms of $E_1$, $m_1$ and $m_2$ ,*

*In the centre of mass frame :*

$$\mathbf{p}_1^2 = \mathbf{p}_2^2 \Rightarrow E_1^2 - m_1^2 = E_2^2 - m_2^2 \Rightarrow E_2 = \left(E_1^2 - m_1^2 + m_2^2\right)^{\frac{1}{2}}$$

$$R_2(E) = \frac{\pi}{(2\pi\hbar)^6} \int \frac{p_1 dE_1}{E_2} \delta\left(E_1 + \left(E_1^2 - m_1^2 + m_2^2\right)^{\frac{1}{2}} - E\right) = \frac{\pi}{(2\pi\hbar)^6} \int \frac{p_1 dE_1}{E_2} \delta(g(E_1))$$

*To integrate over $E_1$ is used:* $\int dE_1 \delta(g(E_1)) = \left|\frac{dg}{dE_1}\right|^{-1}$

$$g(E_1) = E_1 + \left(E_1^2 - m_1^2 + m_2^2\right)^{\frac{1}{2}} - E$$

$$\frac{dg}{dE_1} = 1 + E_1\left(E_1^2 - m_1^2 + m_2^2\right)^{-\frac{1}{2}} = \frac{E_2 + E_1}{E_2} = \frac{E}{E_2} \Rightarrow \left|\frac{dg}{dE_1}\right|^{-1}_{g(E_1)=0} = \frac{E_2}{E}$$

***Two+body Lorenty invariant phase space is:*** $R_2(E) = \frac{\pi}{(2\pi\hbar)^6} \frac{p_1}{E}$

$$E = E_1 + E_2 = \left(p_1^2 + m_1^2\right)^{\frac{1}{2}} + \left(p_2^2 + m_2^2\right)^{\frac{1}{2}} = \left(p_1^2 + m_1^2\right)^{\frac{1}{2}} + \left(p_1^2 + m_2^2\right)^{\frac{1}{2}} \text{ as } p_1^2 = p_2^2$$

$$p_1 = \frac{\left\{\left[E^2 - (m_2 - m_1)^2\right]\left[E^2 - (m_2 + m_1)^2\right]\right\}^{\frac{1}{2}}}{2E}$$

# *Finding n-body phase space recursively*

*N – body phase space in the centre of mass can be rewritten:*

$$R_n(E) = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta\left(\sum_{i=1}^{n} \mathbf{p}_i\right) \delta\left(\sum_{i=1}^{n} E_i - E\right)$$

*as:*

$$R_n(E) = \frac{1}{(2\pi\hbar)^{3n}} \int \frac{d^3 p_n}{2E_n} \int \prod_{i=1}^{n-1} \frac{d^3 p_i}{2E_i} \delta\left(\sum_{i=1}^{n-1} \mathbf{p}_i - (-\mathbf{p}_n)\right) \delta\left(\sum_{i=1}^{n-1} E_i - (E - E_n)\right)$$

*The second integral is the phase space integral for n-1 particles*

*with total momentum –$\mathbf{p}_n$ and total energy ($E$-$E_n$)*

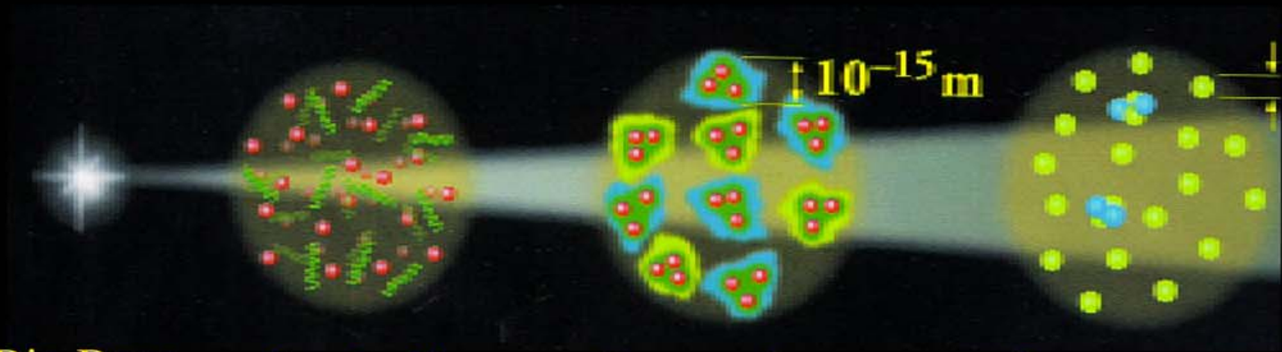
*Lorentz invariance allows this to be rewritten in terms of*

*a system of zero total momentum and energy* $\varepsilon^2 = (E - E_n)^2 - p_n^2$

*Example: let's go to 3-body phase space from 2-body*

$$R_3(E) = \frac{\pi}{(2\pi\hbar)^9} \int \frac{d^3 p_3}{2E_3} \frac{p_1(\varepsilon(E_3))}{\varepsilon} \quad \text{where } \varepsilon^2 = (E - E_3)^2 - p_3^2$$

# History of the Universe




Big Bang

Quark-Gluon Plasma
$10^{13}$K, $10^{-6}$s

Protons& Neutrons
$10^{12}$K, $10^{-4}$s

Low-mass Nuclei
$10^{9}$K, 3 min


Neutral Atoms
4000K, $10^{5}$y

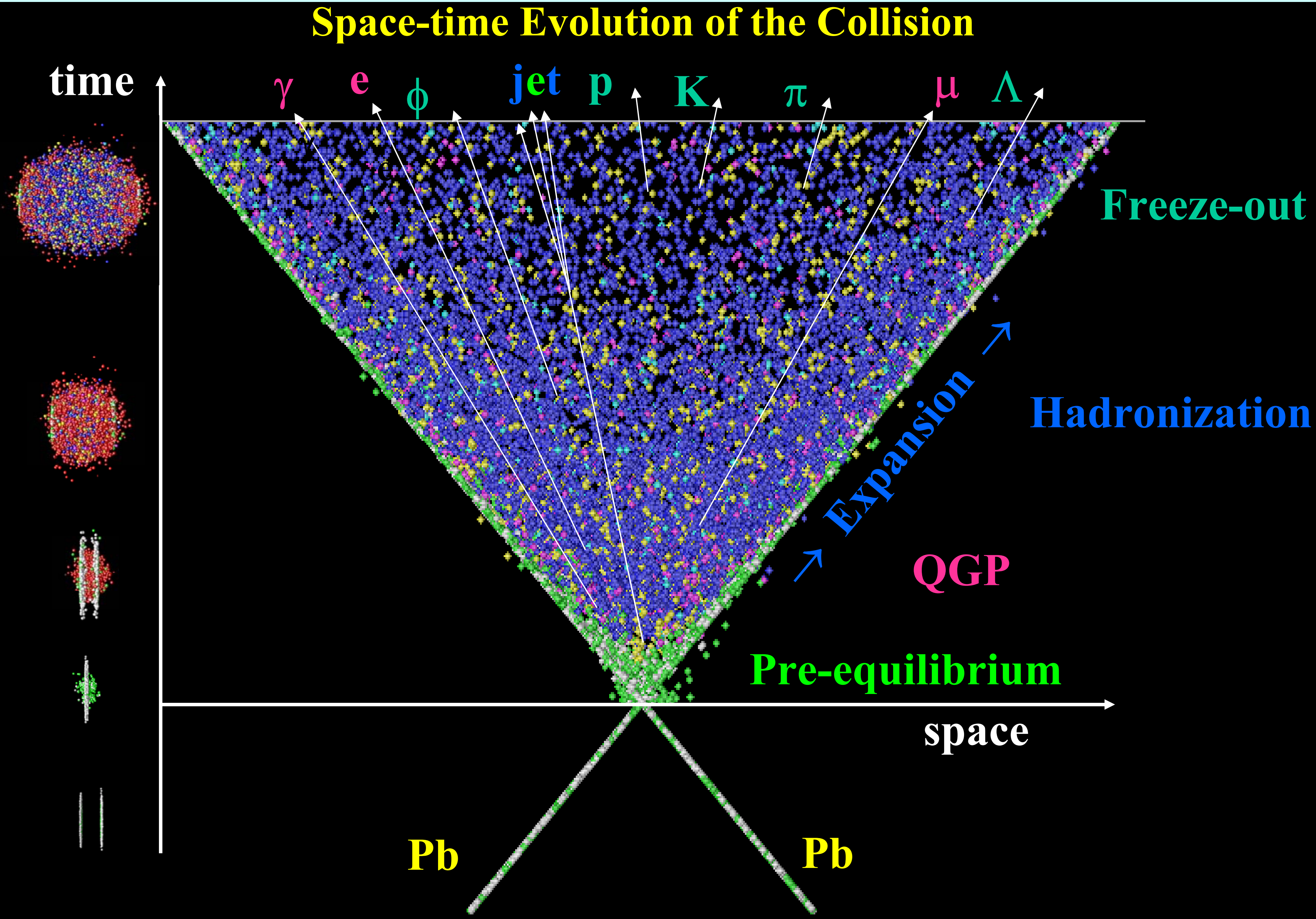
Star Formation
$10^{9}$y

Heavy Elements
$>10^{9}$y

Today

Source: Nuclear Science Wall Chart

# *High energy heavy ion collisions*



# *High energy heavy ion collisions*

Kinematic variables

***Rapidity:*** $$y = \frac{1}{2}\ln(\frac{E + P_Z}{E - P_Z})$$

***Pseudo-rapidity:*** $$\eta = \frac{1}{2}\ln(\frac{P + P_Z}{P - P_Z}) = -\ln(\tan\frac{\theta}{2})$$

***Transverse Momentum:*** $$p_T = \sqrt{p_X^2 + p_Y^2}$$

***Transverse Mass:*** $$m_T = \sqrt{p_T^2 + m_0^2}$$

# *High energy heavy ion collisions*

## *Useful Expressions*

$$E = m_T \cosh y$$

$$p_Z = m_T \sinh y$$

$$\beta_z = \tanh y$$

$$dp_z = E dy$$

***Feymann $x_F$:***

$$x_F = \frac{p_L^*}{p_{\max}^*} = \frac{p_L^*}{\sqrt{s}/2}$$

$$q^2 = (p^i - p^f)^2; Q^2 = -q^2$$

***Bjorken x:***

$$\nu = (E^i - E^f)$$

$$x = \frac{Q^2}{2M\nu}$$

***Light-cone $x_+$:***

$$x_+ = \frac{(E + p_z)}{(E + p_z)_{beam}}$$

# *High energy heavy ion collisions*

## *Invariant Cross Sections*

***Invariant Differential Cross Section:***

$$E\, d^3\sigma/d^3p \rightarrow \frac{d^2\sigma}{2\pi p_T dp_T dy}$$

$$\frac{d^2\sigma}{2\pi m_T dm_T dy}$$

***Invariant Multiplicity Density:***

$$E\, d^3n/d^3p \rightarrow \frac{d^2N}{N_{ev} 2\pi p_T dp_T dy}$$

$$\frac{d^2N}{N_{ev} 2\pi m_T dm_T dy}$$

# High energy heavy ion collisions

## Spectrum Fit

***$m_T$ spectrum:*** $d^2n/(2\pi m_T)dm_T dy$ versus $(m_T - m_0)$

***$p_T$ spectrum:*** $d^2n/(2\pi p_T)dp_T dy$ versus $p_T$

***Boltzmann $m_T$ fit:*** $d^2n/(2\pi m_T)dm_T dy \sim m_T \exp(-m_T \cdot slope)$

*where slope → parameter* **(1/T)**

***Why is this Boltzmann?***

$$d^3n/d^3p \sim \exp(-E/T)$$

***Invariant Multiplicity Density:***

$$Ed^3n/d^3p \sim E \exp(-E/T)$$

$$E = m_T \cosh(y - y_{cm})$$

$$d^2n/(2\pi m_T)dm_T dy \sim m_T \cosh(y-y_{cm}) \exp(-m_T \cosh(y-y_{cm})/T)$$

*Slope depends on rapidity for an isotropic thermal fireball*

$$slope = \cosh(y-y_{cm})/T$$

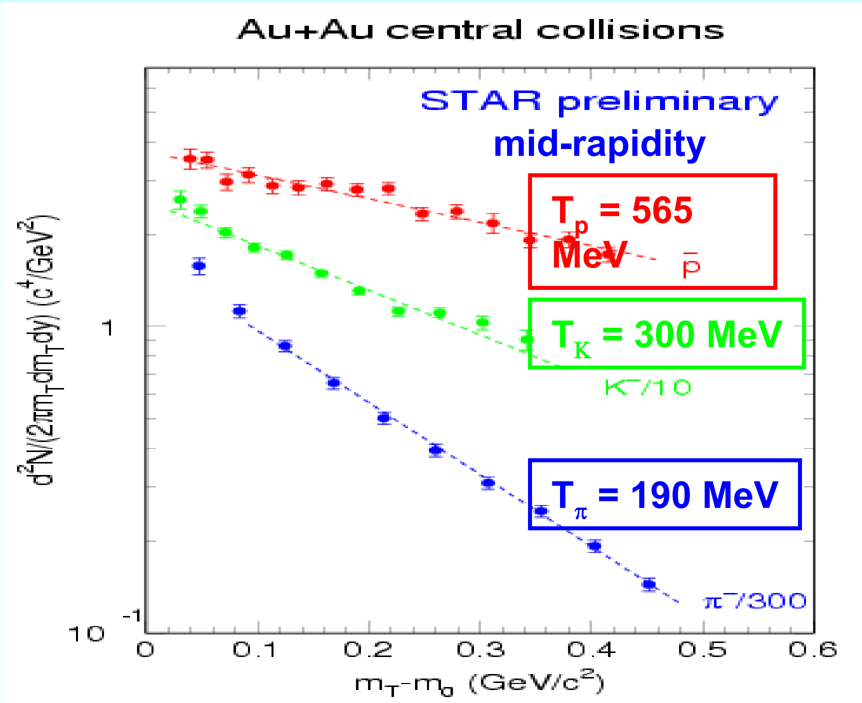
$$dn/dy = \int \left(\frac{d^2n}{2\pi m_T dm_T dy}\right) 2\pi m_T dm_T \approx e^{-\frac{(y-y_{cm})^2}{2\sigma_y^2}}$$

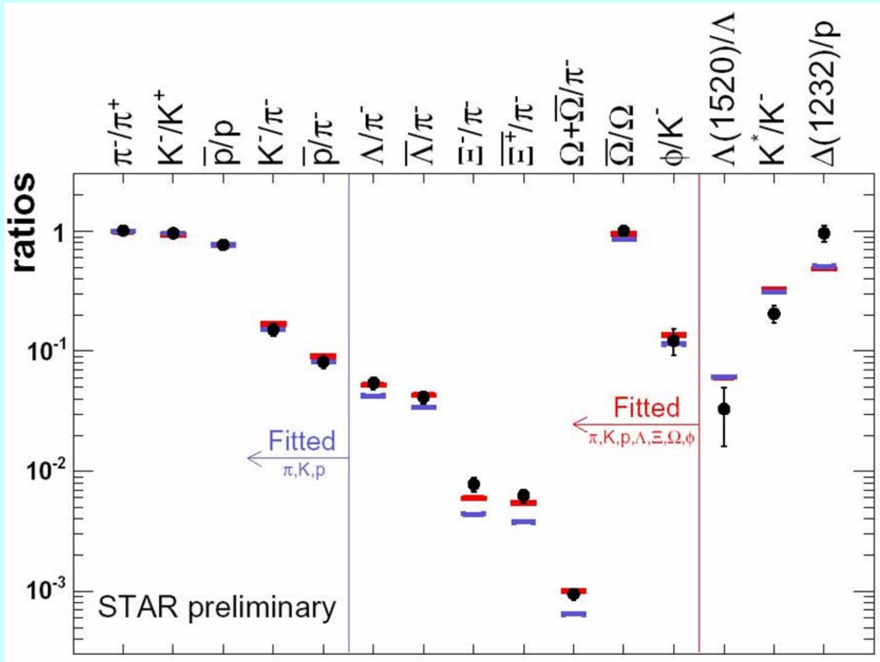
# *High energy heavy ion collisions*

## *Naive Expectations*

*Thermal Isotropic Source → $m_T$ Scaling*

*π, K and p have the same slope parameter $e^{-E/T}$*




- *Data show a large difference among these particles*
- *Unrealistic high temperature*

?

*Statistical models predict $T_{ch}$ ~ 165 MeV*

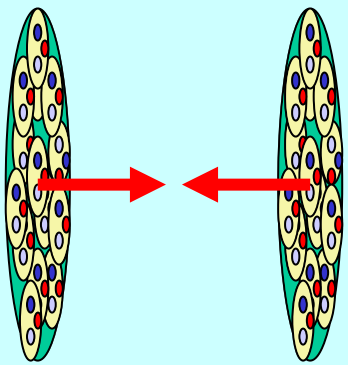
# *High energy heavy ion collisions*

*The invariant momentum spectrum of particles radiated by a thermal source with temperature T is :*

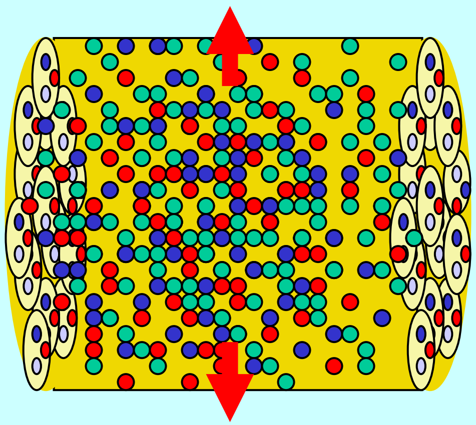
$$E\frac{d^3n}{d^3p} = \frac{dn}{dy\, m_T dm_T\, d\phi} = \frac{gV}{(2\pi)^3} E\, e^{-(E-\mu)/T}$$

*If the quantum statistics is neglected where the particles are decoupled from each other, i.e. where the spectra are computed, $e^{\mu/T}$ can be introduced in the normalization together with V, therefore, integrating over rapidity using modified Bessel function $K_1$ :*

$$\frac{dn}{m_T dm_T} = \frac{V}{2\pi^2} m_T K_1\left(\frac{m_T}{T}\right) \xrightarrow{m_T \gg T} V' \sqrt{m_T}\, e^{-m_T/T}$$



$\Rightarrow$



***Which is the $EdN/d^3p$ for an expanding gas ?***

# *High energy heavy ion collisions*

*Single-particle distribution* **EdN/d³p** *for a relativistic gas described by a distribution function obeying the Boltzmann transport equation.*

*Is the momentum distribution in the center-of-mass frame given by the probability of finding a particle with velocity v times the Lorentz boosted thermal distribution normalized to the total number of particles?*

$$EdN/d^3p = \int dN/d^3v \, \frac{g(\bar{E},\bar{T}(\vec{v}))}{\bar{n}(\bar{T}(\vec{v}))} \, \bar{E} d^3v$$

*where* $\bar{\mathbf{E}}$ *and* $\bar{\mathbf{T}}$ *are the energy and temperature, respectively, in the comoving or local rest frame of the collective motion.*

***This yield the correct number of particles but it is inconsistent with energy conservation***

***The invariant momentum spectrum is given by:***

$$E\frac{d^3n}{d^3p} = \int_\sigma f(x,p)\, p^\lambda d\sigma_\lambda \approx \frac{g}{(2\pi)^3}\int e^{-(u^\nu p_\nu - \mu)/T} p^\lambda d\sigma_\lambda$$

$f(x,p)$ *obeys the Boltzmann equation* $p^\mu \partial_\mu f(x,p) = \Delta\Gamma(x,p)$

$\Delta\Gamma$ *is the rate of change in g due to collision*

# *High energy heavy ion collisions*

*This measures the particle flow through the given hypersurface $\sigma$ as if the virtual particle walls of the fluid cells have suddenly disappeared and the particles are flying isotropic in all directions*

*$\sigma$ defines the borderline between hydrodynamical behaviour and free-streaming particles*

*obviously these are mathematical idealizations*

*In reality the freeze-out hypersurface could be defined by points of the last interaction of each individual particle $\rightarrow$ a thickness of the order of mean free path*

*If the hypersurface $\sigma(r,\phi,\zeta)$ in cylindrical coordinates $0 \le r \le R$, $0 \le \phi \le 2\pi$ and $-Z \le \zeta \le Z$ longitudinal directions*

$$\begin{aligned} \sigma^\mu(r,\phi,\zeta) &= (\, t(\zeta),\, r\cos\phi,\, r\sin\phi,\, z(\zeta)\,) \\ p^\mu d\sigma_\mu &= \left[E\frac{\partial z}{\partial \zeta} - p_L\frac{\partial t}{\partial \zeta}\right] r dr\, d\phi\, d\zeta \end{aligned}$$

$$u^\mu p_\mu \;=\; m_T \cosh(y-\eta)\cosh\rho - p_T \sinh\rho\cos(\phi-\varphi)$$

# *High energy heavy ion collisions*

*Because of azimuthal symmetry we can integrate over $\phi$ making use of the modified Bessel function:*

$$I_0(z) = (2\pi)^{-1} \int_0^{2\pi} e^{z\cos\phi} d\phi$$

$$E\frac{d^3n}{d^3p} = \frac{g}{(2\pi)^2}\int_{-\mathcal{Z}}^{\mathcal{Z}} d\zeta \left[m_T \cosh y \frac{\partial z}{\partial \zeta} - m_T \sinh y \frac{\partial t}{\partial \zeta}\right] \times \int_0^R r\,dr\, \exp\left(-\frac{m_T \cosh\rho \cosh(y-\eta) - \mu}{T}\right) I_0\left(\frac{p_T \sinh\rho}{T}\right)$$

*For the transverse mass spectrum we integrate with the help of another modified Bessel function:*

$$\frac{dn}{m_T dm_T} = \frac{g}{\pi} m_T \int_{-\mathcal{Z}}^{\mathcal{Z}} d\zeta \left[\cosh\eta \frac{\partial z}{\partial \zeta} - \sinh\eta \frac{\partial t}{\partial \zeta}\right] \int_0^R r\,dr\, K_1\left(\frac{m_T\cosh\rho}{T}\right) I_0\left(\frac{p_T \sinh\rho}{T}\right)$$

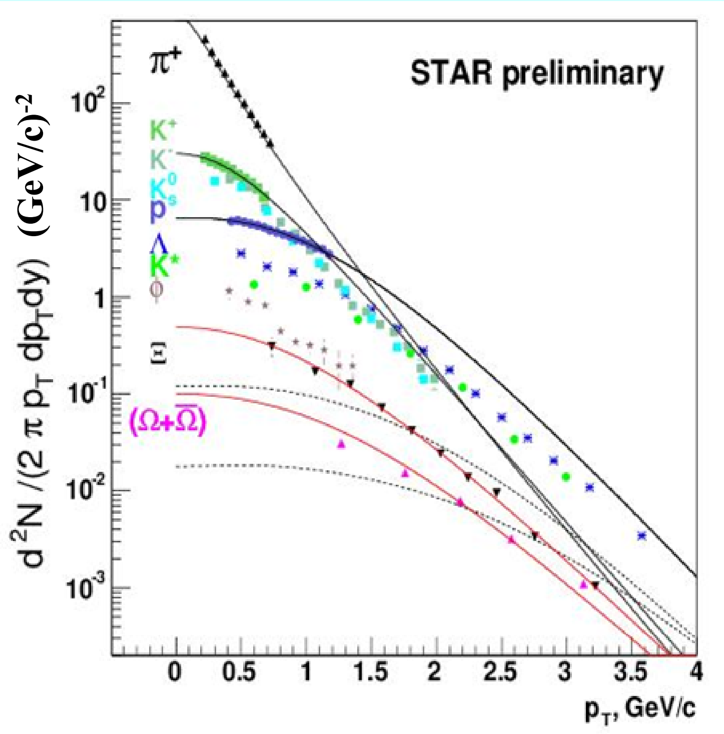
$$= \frac{2g}{\pi} m_T Z_{\hat{t}} \int_0^R r\,dr\, K_1\left(\frac{m_T\cosh\rho}{T}\right) I_0\left(\frac{p_T \sinh\rho}{T}\right)$$

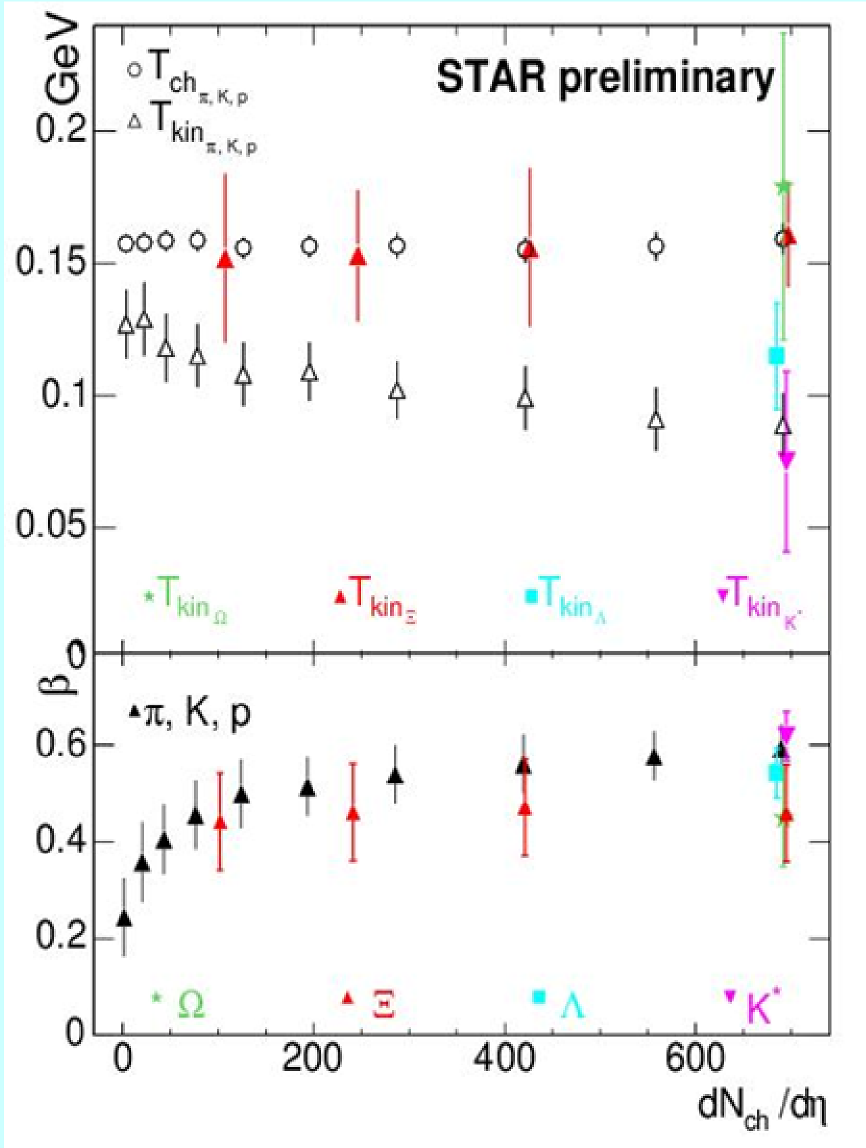
*We can immediately derive the limiting case for large $m_T$ , since $m_T/p_T \rightarrow 1$, $K_0/K_1 \rightarrow 1$ and for finite flow (sinh $\rho > 0$) also $I_1/I_0 \rightarrow 1$:*

$$\lim_{m_T\to\infty} \frac{d}{dm_T} \ln\left(\frac{dn}{m_T dm_T}\right) = \frac{\cosh\rho - \sinh\rho}{T} = -\frac{1}{T}\sqrt{\frac{1-\beta_r}{1+\beta_r}}$$

$$T_{\mathrm{eff}} = T\sqrt{\frac{1+\beta_r}{1-\beta_r}}$$

# *High energy heavy ion collisions*



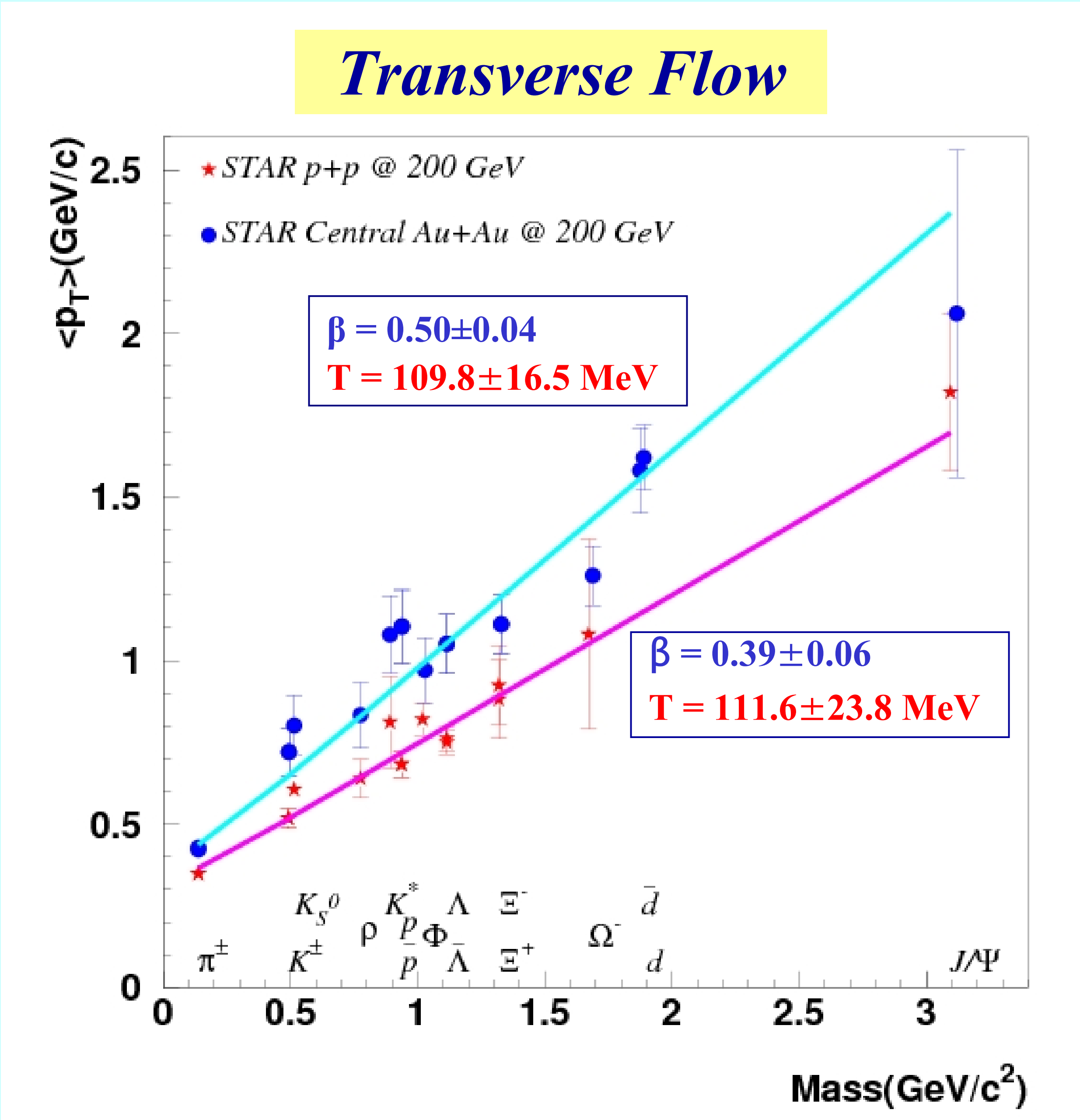




$$\frac{dN}{m_t dm_t} \sim \int_0^R r\,dr\; m_T K_1\left(\frac{m_t \cosh\rho}{T_{f0}}\right) I_0\left(\frac{p_t \sinh\rho}{T_{f0}}\right)$$

$\rho = \tanh^{-1}\beta_r$ $\qquad$ $\beta_r = \beta_s \left(\frac{r}{R}\right)^{\alpha}$ $\alpha = 0.5, 0.7, 1, 2$

**Schnedermann et al nucl-th/9307020**

# High energy heavy ion collisions




$$< p_t >= \frac{\int_0^\infty p_t^2 f(p_t) dp_t}{\int_0^\infty p_t f(p_t) dp_t}$$

$$f(p_t) \sim \int_0^R r dr m_t I_0 \left(\frac{p_t sinh\rho}{T}\right) K_1 \left(\frac{m_t cosh\rho}{T}\right)$$

**$\pi^{\pm}$, $K^{\pm}$, $K^*$, $K_s^0$, p, $\bar{p}$, d, $\bar{d}$**

**β = 0.54±0.04, T = 98.7 ± 19.5 MeV**

# PRELIMINARY

**$\Lambda$, $\bar{\Lambda}$, $\Xi^{\pm}$, $\Omega^-$, J/ψ**

**β = 0.37±<u>0.25</u>, T = 215 ± <u>200</u> MeV ?**

**50% error bars !!!**

**$\Lambda$, $\bar{\Lambda}$, $\Xi^{\pm}$, $\Omega^-$, J/ψ**

**β = 0.37±0.12, T = 215 ± 89 MeV**

M.Petrovici and Amalia Pop
- AIP Conference Proceedings 972(2008)98

# High energy heavy ion collisions

## Transverse Flow

*$f(x,p) \sim [1-(1-q)p^{\mu}U_{\mu}/T]^{1/(1-q)}$ - Tsallis non-extensive statistics*

*$\rightarrow$ Boltzmann for $(q-1) \rightarrow 0$*

**$\rightarrow$ Tsallis Blast Wave:**

$$f(p_t) = m_t \int_{-Y}^{Y} cosh(y)dy \int_{-\pi}^{\pi} d\phi \int_{0}^{R} rdr(1+\frac{q-1}{T}(m_t cosh(y)cosh(\rho)-p_t sinh(\rho)cos(\phi)))^{-1/(q-1)}$$

$$< p_t >= \frac{\int_0^\infty p_t^2 f(p_t)dp_t}{\int_0^\infty p_t f(p_t)dp_t}$$

**A. Lavagno, Phys.Lett. A301(2002)13**
**Z. Tang et al, arXiv:0812.1609 nucl-ex**

**$\pi^{\pm}$, $K^{\pm}$, $K^*$, $K_s^0$, $\bar{p}$, $\Lambda$, $\bar{\Lambda}$, $\Xi^{\pm}$, $\Omega^-$, d, $\bar{d}$, J/$\psi$**

| System | p + p | p + p | Au + Au | Au + Au |
|---|---|---|---|---|
| Model | BGBW | TBW | BGBW | TBW |
| T [MeV] | 111.6±23.8 | 78.86±10.13 | 109.8±16.5 | 86.8 ±1.54 |
| $\beta$ | 0.39±0.06 | 0.027±0.10 | 0.50±0.04 | 0.48±0.04 |
| q | 1.0 | 1.0874 | 1.0 | 1.0247 |

**$\pi^{\pm}$, $K^{\pm}$, $K^*$, p, $\bar{p}$, d, $\bar{d}$**

| | Au + Au | Au + Au |
|---|---|---|
| | BGBW | TBW |
| T [MeV] | 98.7±19.5 | 79.05±0.04 |
| $\beta$ | 0.54±0.04 | 0.53±0.0005 |
| q | 1.0 | 1.0175 |

**$\Lambda$, $\bar{\Lambda}$, $\Xi^{\pm}$, $\Omega^-$, J/$\psi$**

| | Au + Au |
|---|---|
| | TBW |
| T [MeV] | 198.0±7.6 |
| $\beta$ | 0.32±0.012 |
| q | 1.0247 |

| | Au + Au | Au + Au |
|---|---|---|
| | BGBW | TBW |
| T [MeV] | 215.5±89.2 | 200.5±0.1 |
| $\beta$ | 0.37±0.12 | 0.36 |
| q | 1.0 | 1.021 |

**M.Petrovici and Amalia Pop**
**arXiv:0904.3666**